Lecture 03
Kinematics & Jacobian

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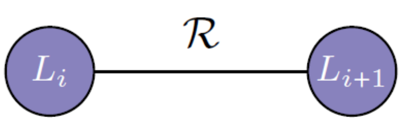
Forward Kinematics

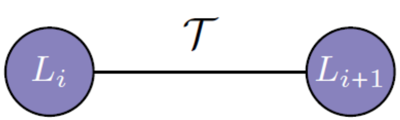
General Procedure

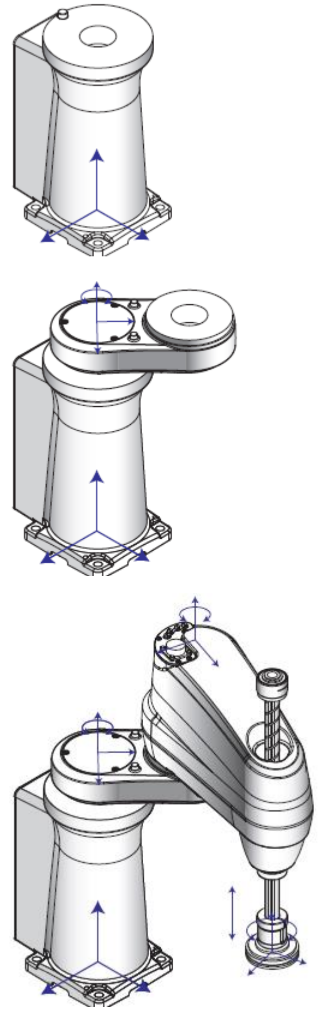
Adept Cobra i600 (SCARA)

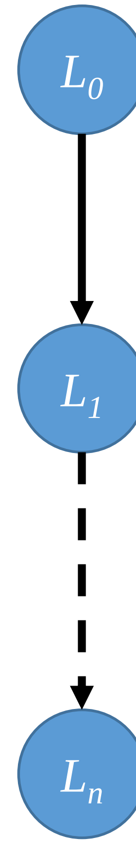
revolute joint $S^1 \mapsto SO(2)$



prismatic joint $\mathbb{R} \mapsto T(1)$







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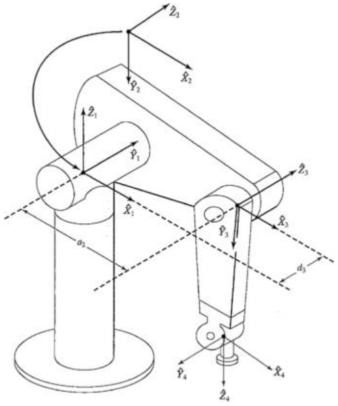
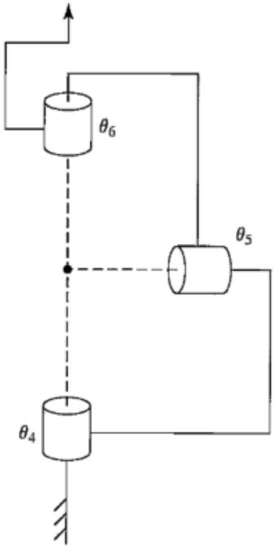
Forward Kinematics

Example with PUMA 560



i	$\alpha_i - 1$	$a_i - 1$	d_i	θi
1	0	0	0	θ_1
2	$-90°$	0	0	θ_2
3	0	a_2	d_3	θ_3
4	$-90°$	a_3	d_4	θ_4
5	$90°$	0	0	θ_5
6	$-90°$	0	0	θ_6



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$$ {}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^2_3T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^3_4T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^4_5T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^5_6T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

$$ {}^0_6T = {}^0_1T \, {}^1_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ r_{11} = c_1[c_{23}(c_4c_5c_6 - s_4s_5) - s_{23}s_5c_5] + s_1(s_4c_5c_6 + c_4s_6), $$
$$ r_{21} = s_1[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6 - c_1(s_4c_5c_6 + c_4s_6), $$
$$ r_{31} = -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6, $$

$$ r_{12} = c_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] + s_1(c_4c_6 - s_4c_5s_6), $$
$$ r_{22} = s_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] - c_1(c_4c_6 - s_4c_5s_6), $$
$$ r_{32} = -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6, $$

$$ r_{13} = -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5, $$
$$ r_{23} = -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5, $$
$$ r_{33} = s_{23}c_4s_5 - c_{23}c_5, $$

$$ p_x = c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1, $$
$$ p_y = s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] + d_3c_1, $$
$$ p_z = -a_3s_{23} - a_2s_2 - d_4c_{23}. $$

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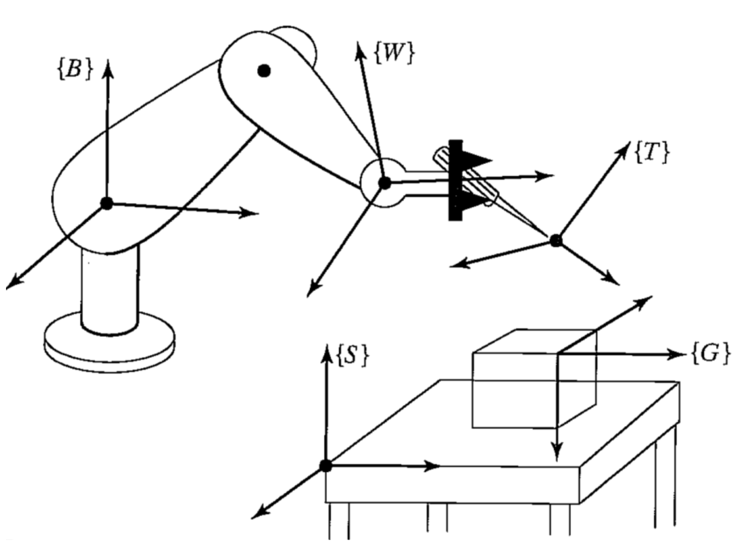
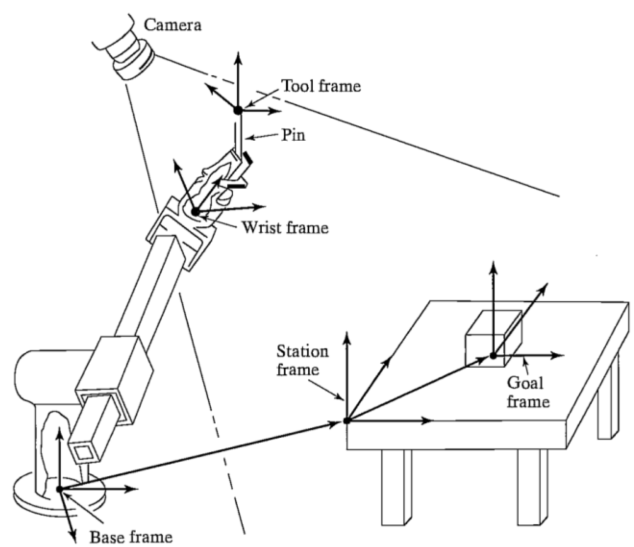
Inverse Kinematics

$$\{B\} \leftarrow [\theta_1\ \theta_2 \ldots\ \theta_n] \leftarrow \{W\} \leftarrow \mathbf{\{T\}} \rightarrow \{G\} \rightarrow \{S\}$$

- Solving the problem of finding the required joint angles $[\theta_1\ \theta_2 \ldots\ \theta_n]$ to place the tool frame, {T}, relative to the station frame, {S}
  - Find the wrist frame, {W}, relative to the base frame, {B}
  - The inverse kinematics are used to solve for the joint angles





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Solvability

Solving a ***nonlinear*** set of equations

- Given ${}^0_6T$ as sixteen numeric values (four of which are trivial), solve for the six joint angles $[\theta_1\ \theta_2 \ldots\ \theta_6]$.
  - 12 equations and six unknowns
  - Rotation Matrix
    - 3 out 9 equations are independent
  - Position Vector
    - 3 independent

$$ {}^0_6T = {}^0_1T\ {}^1_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ r_{11} = c_1[c_{23}(c_4c_5c_6 - s_4s_5) - s_{23}s_5c_5] + s_1(s_4c_5c_6 + c_4s_6), $$
$$ r_{21} = s_1[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6 - c_1(s_4c_5c_6 + c_4s_6), $$
$$ r_{31} = -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6, $$

$$ r_{12} = c_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] + s_1(c_4c_6 - s_4c_5s_6), $$
$$ r_{22} = s_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] - c_1(c_4c_6 - s_4c_5s_6), $$
$$ r_{32} = -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6, $$

$$ r_{13} = -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5, $$
$$ r_{23} = -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5, $$
$$ r_{33} = s_{23}c_4s_5 - c_{23}c_5, $$

$$ p_x = c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1, $$
$$ p_y = s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] + d_3c_1, $$
$$ p_z = -a_3s_{23} - a_2s_2 - d_4c_{23}. $$

A Challenging Task

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Workspace

Volume of space that the end-effector of the manipulator can reach

(a) Workspace calculation:

$$ {}^{B}_{W}T = \begin{bmatrix} c_\phi & -s_\phi & 0.0 & x \\ s_\phi & c_\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ g = (x, y, \phi) $$
$$ x = l_1 c_1 + l_2 c_{12} + l_3 c_{123} $$
$$ y = l_1 s_1 + l_2 s_{12} + l_3 s_{123} $$
$$ \phi = \theta_1 + \theta_2 + \theta_3 $$

(b) Construction of Workspace

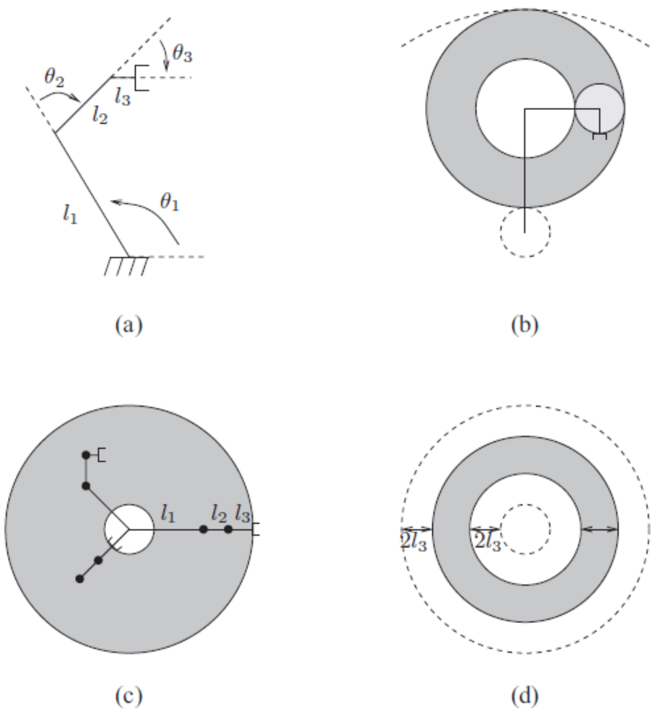
(c) Reachable Workspace

- Volume of space that the robot can reach in at least one orientation

(d) Dexterous Workspace

- Volume of space that the robot end-effector can reach with all orientations

Workspace Example:
A planar serial 3-bar linkage



The Tool Frame also needs to be considered in applications

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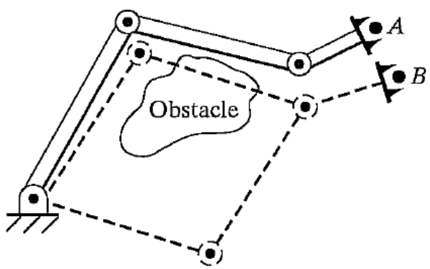
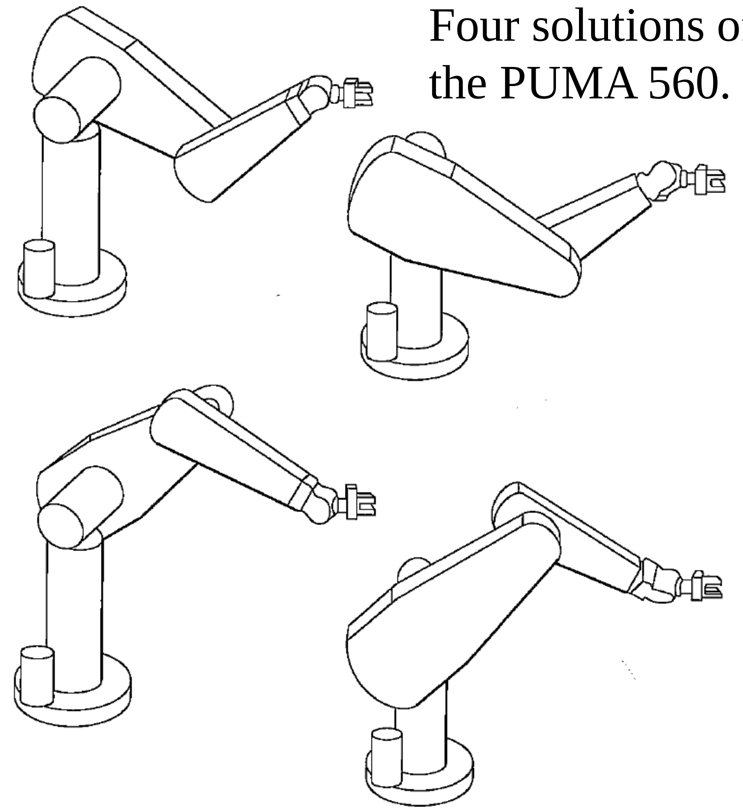
Multiple Solutions

The robot needs to make a choice, sometimes difficult

One of the two possible solutions to reach point B causes a collision.



Four solutions of the PUMA 560.



How the maximum number of solutions is related to how many of the link length parameters (the a_i) are zero.

- The more that are nonzero, the bigger is the maximum number of solutions.
- For a completely general rotary-jointed manipulator with six degrees of freedom, there are up to sixteen solutions possible.

a_i	Number of solutions
$a_1 = a_3 = a_5 = 0$	≤ 4
$a_3 = a_5 = 0$	≤ 8
$a_3 = 0$	≤ 16
All $a_i \neq 0$	≤ 16

[1] B. Roth, J. Rastegar, and V. Scheinman, "On the Design of Computer Controlled Manipulators," *On the Theory and Practice of Robots and Manipulators*, Vol. 1, First CISM-IFToMM Symposium, September 1973, pp. 93–113.

[6] L. Tsai and A. Morgan, "Solving the Kinematics of the Most General Six- and Five-degree-of-freedom Manipulators by Continuation Methods," Paper 84-DET-20, ASME Mechanisms Conference, Boston, October 7–10, 1984.

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Method of Solution

There are no general algorithms that may be employed to solve a set of nonlinear equations

- Closed-form solutions vs. Numerical solutions
  - A closed solution method is based on analytic expressions or on the solution of a polynomial of degree 4 or less, such that noniterative calculations suffice to arrive at a solution.
- ***algebraic*** and ***geometric*** methods for a closed solution
  - The bottom-line is that *all systems with revolute and prismatic joints having a total of six degrees of freedom in a single series chain are solvable*
    - However, this general solution is a numerical one.
    - Only in special cases can robots with six degrees of freedom be solved analytically
  - The ***"trick"*** to design a ***closed-form-solvable*** robot
    - Intersecting joint axes or with twist angles equals to 0 or ±90 degrees

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Algebraic Solution

Consider a three-link planar manipulator

- All attainable goals must lie in the subspace

$$ {}^{B}_{W}T = \begin{bmatrix} c_\phi & -s_\phi & 0.0 & x \\ s_\phi & c_\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0.0 & l_1s_1 + l_2s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$c_\phi = c_{123},$$

$$s_\phi = s_{123},$$

$$x = l_1c_1 + l_2c_{12},$$

$$y = l_1s_1 + l_2s_{12}.$$

$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1l_2c_2,$$

$$c_{12} = c_1c_2 - s_1s_2,$$

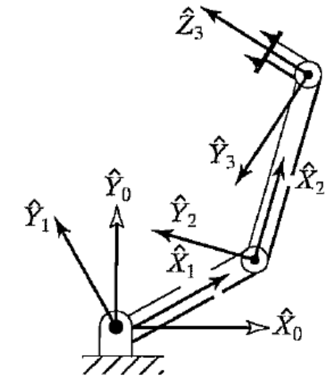
$$s_{12} = c_1s_2 + s_1c_2.$$

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}.$$

Check if a solution exist within [-1,1]

$$s_2 = \pm\sqrt{1 - c_2^2}.$$

$$\theta_2 = \text{Atan2}(s_2, c_2).$$



i	$\alpha_i - 1$	$\alpha_i - 1$	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

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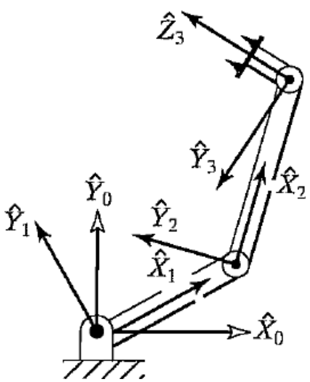
Algebraic Solution

Solving the rest joint angles

Algebraic solution by reduction to polynomial

$$u = \tan\frac{\theta}{2},$$
$$\cos\theta = \frac{1-u^2}{1+u^2},$$
$$\sin\theta = \frac{2u}{1+u^2}.$$

$$c_\phi = c_{123},$$
$$s_\phi = s_{123},$$
$$x = l_1 c_1 + l_2 c_{12},$$
$$y = l_1 s_1 + l_2 s_{12}.$$



$$x = k_1 c_1 - k_2 s_1, \quad k_1 = l_1 + l_2 c_2,$$
$$y = k_1 s_1 + k_2 c_1, \quad k_2 = l_2 s_2.$$

$$r = +\sqrt{k_1^2 + k_2^2} \qquad k_1 = r\cos\gamma,$$
$$\gamma = \text{Atan2}(k_2, k_1), \qquad k_2 = r\sin\gamma.$$

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1,$$
$$\frac{y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1,$$

$$\cos(\gamma+\theta_1) = \frac{x}{r},$$
$$\sin(\gamma+\theta_1) = \frac{y}{r}.$$

$$\gamma + \theta_1 = \text{Atan2}\left(\frac{y}{r}, \frac{x}{r}\right) = \text{Atan2}(y, x),$$

$$\theta_1 = \text{Atan2}(y, x) - \text{Atan2}(k_2, k_1).$$
$$\theta_2 = \text{Atan2}(s_2, c_2).$$

$$\theta_1 + \theta_2 + \theta_3 = \text{Atan2}(s_\phi, c_\phi) = \phi.$$

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Pieper's Solution When 3 Axes Intersect

Also include the case with 3 consecutive parallel axes, as they meet at the point at infinity

- When the last three axes intersect, the origins of link frames {4}, {5}, and {6} are all located at this point of intersection.

$$ {}^0P_{4ORG} = {}^0_1T\, {}^1_2T\, {}^2_3T\, {}^3P_{4ORG} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, $$

$$ f_1 = a_3 c_3 + d_4 s\alpha_3 s_3 + a_2, $$
$$ f_2 = a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2, $$
$$ f_3 = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2. $$

$$ {}^0P_{4ORG} = {}^0_1T\, {}^1_2T\, {}^2_3T \begin{bmatrix} a_3 \\ -d_4 s\alpha_3 \\ d_4 c\alpha_3 \\ 1 \end{bmatrix} = {}^0_1T\, {}^1_2T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} $$

$$ \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 1 \end{bmatrix} = {}^2_3T \begin{bmatrix} a_3 \\ -d_4 s\alpha_3 \\ d_4 c\alpha_3 \\ 1 \end{bmatrix} $$

$$ {}^0P_{4ORG} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix} $$

$$ g_1 = c_2 f_1 - s_2 f_2 + a_1, $$
$$ g_2 = s_2 c\alpha_1 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1, $$
$$ g_3 = s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1. $$

the squared magnitude of ${}^0P_{4ORG}$

$$ r = g_1^2 + g_2^2 + g_3^2; $$

$$ r = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2 d_2 f_3 + 2 a_1 (c_2 f_1 - s_2 f_2). $$

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Solving for $\theta_1\ \theta_2\ \theta_3$

$$u = \tan\frac{\theta}{2},$$
$$\cos\theta = \frac{1-u^2}{1+u^2},$$
$$\sin\theta = \frac{2u}{1+u^2}.$$

(4.35)

Pieper's Solution When 3 Axes Intersect

dependence on θ_1 has been eliminated

dependence on θ_2 takes a simple form

(4.50)

$$r = (k_1 c_2 + k_2 s_2) 2a_1 + k_3,$$
$$z = (k_1 s_2 - k_2 c_2) s\alpha_1 + k_4,$$

$$r = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 + 2a_1(c_2 f_1 - s_2 f_2).$$
$$g_3 = s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1.$$

$$k_1 = f_1,$$
$$k_2 = -f_2,$$
$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3,$$
$$k_4 = f_3 c\alpha_1 + d_2 c\alpha_1.$$

$$f_1^2 + f_2^2 + f_3^2 = a_3^2 + d_4^2 + d_3^2 + a_2^2 + 2d_4 d_3 c\alpha_3 + 2a_2 a_3 c_3 + 2a_2 d_4 s\alpha_3 s_3.$$

consider the solution for θ_3

Then, we can solve for θ_1 and θ_2 using (4.50)

1. If $a_1 = 0$, then we have $r = k_3$, where r is known. The right-hand side (k_3) is a function of θ_3 only. After the substitution (4.35), a quadratic equation in $\tan\frac{\theta_3}{2}$ may be solved for θ_3.
2. If $s\alpha_1 = 0$, then we have $z = k_4$, where z is known. Again, after substituting via (4.35), a quadratic equation arises that can be solved for θ_3.
3. Otherwise, eliminate s_2 and c_2 from (4.50) to obtain

$$\frac{(r-k_3)^2}{4a_1^2} + \frac{(z-k_4)^2}{s^2\alpha_1} = k_1^2 + k_2^2. \qquad (4.52)$$

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Solving for $\theta_4\ \theta_5\ \theta_6$

Pieper's Solution When 3 Axes Intersect

- These axes intersect, so these joint angles affect the orientation of only the last link.
  - The rotation portion of the specified goal is enough for solution
  - **One can use exactly the Z-Y-Z Euler angle solution to solve**

- Set one joint to be zero, then solve the other two, finally get back to the joint set to zero
  - We can compute ${}^0_4R|_{\theta_4=0}$
    - the orientation of link frame {4} relative to the base frame when $\theta_4 = 0$

$$ {}^4_6R|_{\theta_4=0} = {}^0_4R^{-1}|_{\theta_4=0}\, {}^0_6R. \Rightarrow \theta_5\ \theta_6 \Rightarrow \theta_4 $$

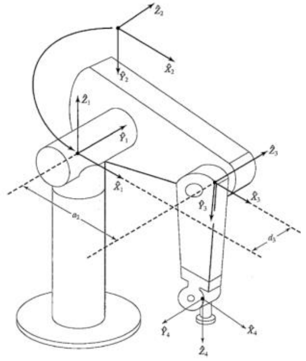
- There are always two solutions for these last three joints,
  - So the total number of solutions for the manipulator will be twice the number found for the first three joints.

Examples Of Inverse Kinematics

The Unimation PUMA 560

$$ {}^0_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = {}^0_1T(\theta_1){}^1_2T(\theta_2){}^2_3T(\theta_3){}^3_4T(\theta_4){}^4_5T(\theta_5){}^5_6T(\theta_6) $$



$$
\begin{aligned}
{}^1r_{11} &= c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6,\\
{}^1r_{21} &= -s_4c_5c_6 - c_4s_6,\\
{}^1r_{31} &= -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6,\\
{}^1r_{12} &= -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6,\\
{}^1r_{22} &= s_4c_5s_6 - c_4c_6,\\
{}^1r_{32} &= s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6,\\
{}^1r_{13} &= -c_{23}c_4s_5 - s_{23}c_5,\\
{}^1r_{23} &= s_4s_5,\\
{}^1r_{33} &= s_{23}c_4s_5 - c_{23}c_5,\\
{}^1p_x &= a_2c_2 + a_3c_{23} - d_4s_{23},\\
{}^1p_y &= d_3,\\
{}^1p_x &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
$$

$$ \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T = \begin{bmatrix} {}^1r_{11} & {}^1r_{12} & {}^1r_{13} & {}^1p_x \\ {}^1r_{21} & {}^1r_{22} & {}^1r_{23} & {}^1p_y \\ {}^1r_{31} & {}^1r_{32} & {}^1r_{33} & {}^1p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ -s_1p_x + c_1p_y = d_3. $$

$$ \theta_1 = \text{Atan2}(p_y, p_x) - \text{Atan2}\left(d_3, \pm\sqrt{p_x^2 + p_y^2 - d_3^2}\right) $$

2 solutions

$$ c_1p_x + s_1p_y = a_3c_{23} - d_4s_{23} + a_2c_2, $$
$$ -p_x = a_3s_{23} + d_4c_{23} + a_2s_2. $$

$$ \theta_3 = \text{Atan2}(a_3, d_4) - \text{Atan2}(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}). $$

2 solutions

$$ K = \frac{p_x^2 + p_y^2 + p_x^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2}. $$

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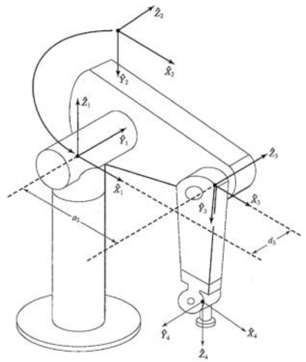
Examples Of Inverse Kinematics

The Unimation PUMA 560

$$[{}^0_3T(\theta_2)]^{-1}\,{}^0_6T = {}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6),$$



$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^3_6T = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 & a_3 \\ s_5c_6 & -s_5s_6 & c_5 & d_4 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$c_1c_{23}p_x + s_1c_{23}p_y - s_{23}p_z - a_2c_3 = a_3,$$
$$-c_1s_{23}p_x - s_1s_{23}p_y - c_{23}p_z + a_2s_3 = d_4.$$

$$\theta_{23} = \text{Atan2}[(-a_3 - a_2c_3)p_z - (c_1p_x + s_1p_y)(d_4 - a_2s_3),\ (a_2s_3 - d_4)p_z - (a_3 + a_2c_3)(c_1p_x + s_1p_y)].$$

$\theta_2 = \theta_{23} - \theta_3$, 2 solutions corresponding to θ_3

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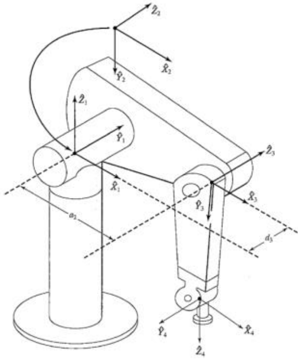
Examples Of Inverse Kinematics

The Unimation PUMA 560



$$[{}^0_3T(\theta_2)]^{-1}{}^0_6T = {}^3_4T(\theta_4){}^4_5T(\theta_5){}^5_6T(\theta_6),$$

Now the entire left side is known

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^3_6T = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 & a_3 \\ s_5c_6 & -s_5s_6 & c_5 & d_4 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{13}c_1c_{23} + r_{23}s_1c_{23} - r_{33}s_{23} = -c_4s_5,$$
$$-r_{13}s_1 + r_{23}c_1 = s_4s_5.$$

As long as $s_5 \neq 0$, we can solve for θ_4 as

$$\theta_4 = \text{Atan2}(-r_{13}s_1 + r_{23}c_1, -r_{13}c_1c_{23} - r_{23}s_1c_{23} + r_{33}s_{23}).$$

When $\theta_5 = 0$, the manipulator is in a singular configuration in which joint axes 4 and 6 line up and cause the same motion of the last link of the robot.

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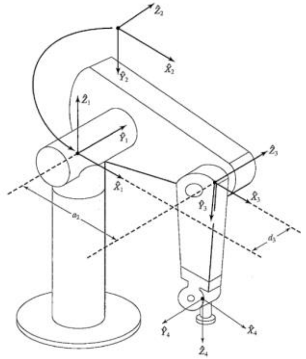
Examples Of Inverse Kinematics

The Unimation PUMA 560

$$[{}^0_4T(\theta_4)]^{-1}\,{}^0_6T = {}^4_5T(\theta_5)\,{}^5_6T(\theta_6),$$



Another 4 flipped wrist solutions

$$\theta_4' = \theta_4 + 180^\circ,$$
$$\theta_5' = -\theta_5,$$
$$\theta_6' = \theta_6 + 180^\circ.$$

A total of 8 solutions

$$\begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & s_1c_{23}c_4 - c_1s_4 & -s_{23}c_4 & -a_2c_3c_4 + d_3s_4 - a_3c_4 \\ -c_1c_{23}s_4 + s_1c_4 & -s_1c_{23}s_4 - c_1c_4 & s_{23}s_4 & a_2c_3s_4 + d_3c_4 + a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_5c_6 & -c_5s_6 & -s_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ s_5c_6 & -s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{13}(c_1c_{23}c_4 + s_1s_4) + r_{23}(s_1c_{23}c_4 - c_1s_4) - r_{33}(s_{23}c_4) = -s_5,$$
$$r_{13}(-c_1s_{23}) + r_{23}(-s_1s_{23}) + r_{33}(-c_{23}) = c_5.$$

$$\theta_5 = \text{Atan2}(s_5, c_5),$$

4 solutions corresponding to $\theta_1\ \theta_3$

$$({}^0_5T)^{-1}\,{}^0_6T = {}^5_6T(\theta_6).$$

$$\theta_6 = \text{Atan2}(s_6, c_6),$$

$$s_6 = -r_{11}(c_1c_{23}s_4 - s_1c_4) - r_{21}(s_1c_{23}s_4 + c_1c_4) + r_{31}(s_{23}s_4),$$
$$c_6 = r_{11}[(c_1c_{23}c_4 + s_1s_4)c_5 - c_1s_{23}s_5] + r_{21}[(s_1c_{23}c_4 - c_1s_4)c_5 - s_1s_{23}s_5] - r_{31}(s_{23}c_4c_5 + c_{23}s_5).$$

The Standard Frames

Classical Methods vs. Learning Methods

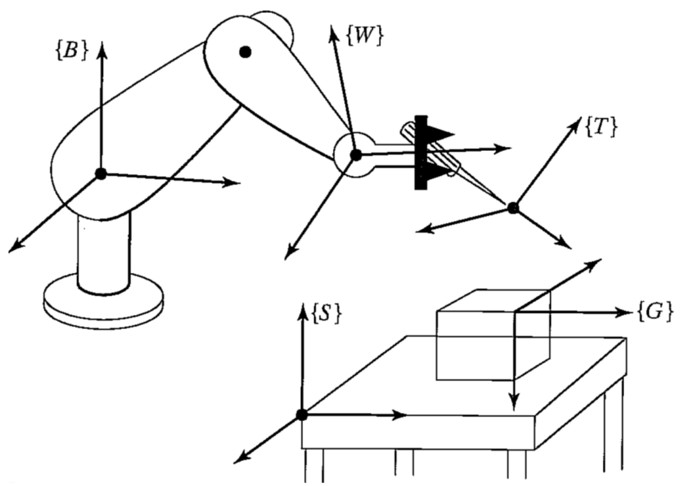
- User Specified Frames {S}, {B}, {T}
  - {S} w.r.t. {B} and {T} w.r.t. {W}
- The “trick” of finding {G}
  - **Classical** method: User Defined
  - **Learning** method: System Learns

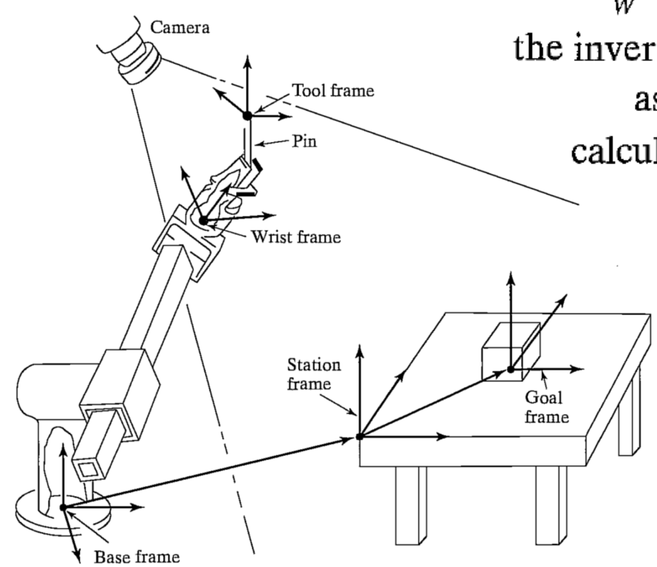
The robot system calculates a series of joint angles to move the joints through in order that the tool frame will move from its initial location in a smooth manner until {T} = {G} at the end of motion.

- Inverse Kinematics
- Motion Planning

$$ {}^{B}_{W}T = {}^{B}_{S}T \; {}^{S}_{T}T \; {}^{W}_{T}T^{-1} $$

the inverse kinematics take ${}^{B}_{W}T$ as an input and calculate θ_1 through θ_n





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Repeatability, Accuracy & Computation

Practical Issues to be Considered

- **Repeatability**: How precisely a manipulator can return to a taught point
  - A taught point: One that the manipulator is moved to physically, and then the joint position sensors are read and the joint angles stored.

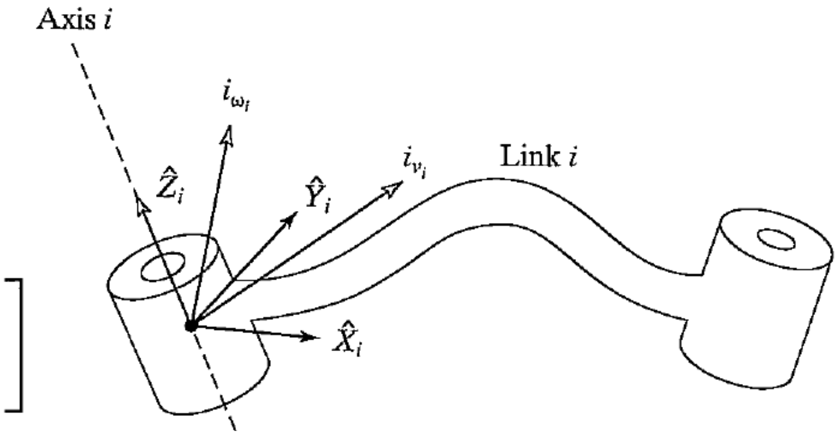
- **Accuracy**: The precision with which a computed point can be attained
  - A computed point: One in its workspace to which it has perhaps never gone before.

- **Computation**: Inverse Kinematics of a manipulator usually needs to be calculated at a fairly high rates, i.e. 30 Hz or faster
  - Numerical solution is usually computationally savvy, closed-form is always preferred
  - A table-lookup Atan2 routine can be used to attain higher speeds (no need to compute)
  - Optimized computation structure to find **JUST THE ONE** instead of **ALL** solutions
  - Usually the 1^{st} solutions takes the most computation and the rest can be obtained by summing and differencing angles, subtracting π, and so on

Motion of the links of a robot

Velocity "Propagation" From Link To Link

- Linear velocity is associated with a point (i.e. origin of the link frame), but angular velocity is associated with a body

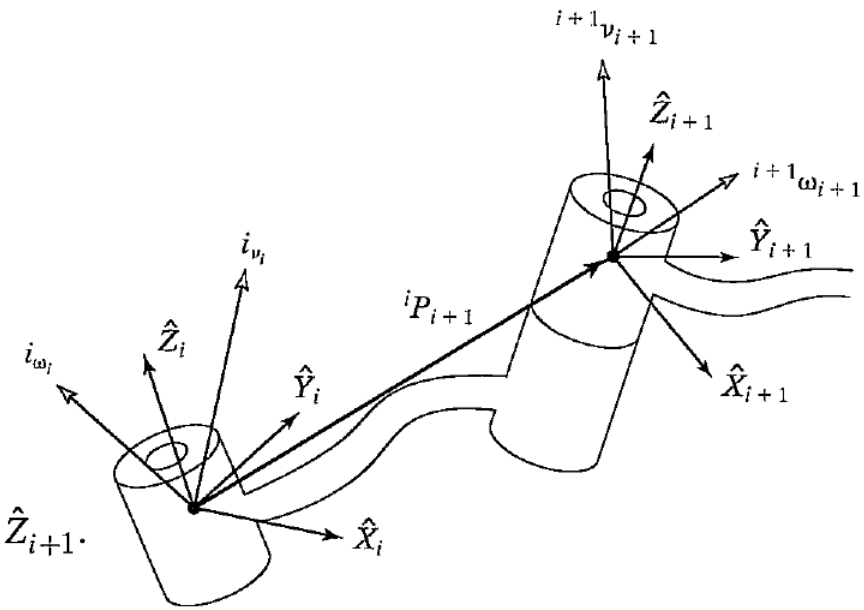


angular velocity

$$ {}^{i}\omega_{i+1} = {}^{i}\omega_i + {}^{i}_{i+1}R\, \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}. $$

$$ \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} $$



linear velocity

$$ {}^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1} $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) $$

For prismatic joint

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i, $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + \dot{d}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}. $$

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Example of a Two-link Manipulator

Calculate the velocity of the tip of the arm as a function of joint rates.
Give the answer in two forms—in terms of frame {3} and also in terms of frame {0}.

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

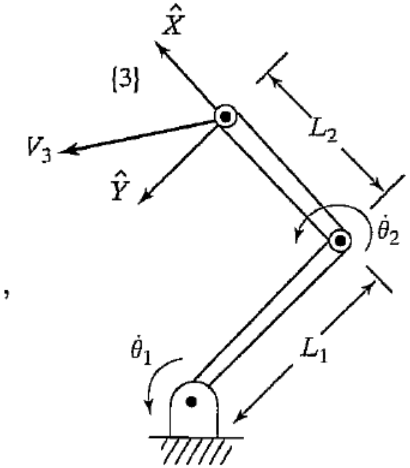
$$ {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

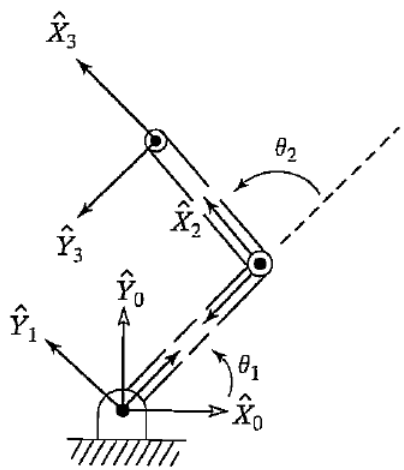
$$ {}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix}, \quad {}^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 \end{bmatrix}, $$

$$ {}^1v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^2v_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_1\dot\theta_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 s_2 \dot\theta_1 \\ l_1 c_2 \dot\theta_1 \\ 0 \end{bmatrix}, $$

$$ {}^3\omega_3 = {}^2\omega_2, $$

$$ {}^3v_3 = \begin{bmatrix} l_1 s_2 \dot\theta_1 \\ l_1 c_2 \dot\theta_1 + l_2(\dot\theta_1 + \dot\theta_2) \\ 0 \end{bmatrix}. $$





$$ {}^0_3R = {}^0_1R \ {}^1_2R \ {}^2_3R = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0v_3 = \begin{bmatrix} -l_1 s_1 \dot\theta_1 - l_2 s_{12}(\dot\theta_1 + \dot\theta_2) \\ l_1 c_1 \dot\theta_1 + l_2 c_{12}(\dot\theta_1 + \dot\theta_2) \\ 0 \end{bmatrix} $$

Jacobian

Adding static forces to the “equations”

$$
\begin{aligned}
y_1 &= f_1(x_1, x_2, x_3, x_4, x_5, x_6),\\
y_2 &= f_2(x_1, x_2, x_3, x_4, x_5, x_6),\\
&\vdots\\
y_6 &= f_6(x_1, x_2, x_3, x_4, x_5, x_6).
\end{aligned}
$$

$$
\begin{aligned}
\delta y_1 &= \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6}\delta x_6,\\
\delta y_2 &= \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_2}{\partial x_6}\delta x_6,\\
&\vdots\\
\delta y_6 &= \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6}\delta x_6,
\end{aligned}
$$

Jacobian might also be found by directly differentiating the kinematic equations of the mechanism

$$Y = F(X).$$

$$\delta Y = \frac{\partial F}{\partial X}\delta X.$$

$$\delta Y = J(X)\delta X.$$

$$\dot{Y} = J(X)\dot{X}.$$

In the field of robotics, we generally use Jacobians that relate joint velocities to Cartesian velocities of the tip of the arm

- The number of **rows** equals the number of **degrees of freedom** in the Cartesian space being considered.
- The number of **columns** in a Jacobian is equal to the **number of joints** of the manipulator.

a vector of Cartesian velocities → $^0\nu = {}^0J(\Theta)\dot{\Theta}$ ← the vector of joint angles of the manipulator

$$^0\nu = \begin{bmatrix} ^0 v \\ ^0\omega \end{bmatrix}$$

Singularities

Is the Jacobian invertible for all values of Θ? If not, where is it not invertible?

- Invertibility problem of linear transformation

$$\dot{\Theta} = J^{-1}(\Theta)\nu.$$

At a singular point, the inverse Jacobian blows up!

- This results in joint rates approaching infinity as the singularity is approached

  - We could calculate the necessary joint rates at each instant along the path

- Singularities of the Mechanism
  - Values of Θ where the Jacobian becomes singular
    - All manipulators have singularities at the boundary of their workspace,
    - Most have loci of singularities inside their workspace

1. **Workspace-boundary singularities** occur when the manipulator is fully stretched out or folded back on itself in such a way that the end-effector is at or very near the boundary of the workspace.

2. **Workspace-interior singularities** occur away from the workspace boundary; they generally are caused by a lining up of two or more joint axes.

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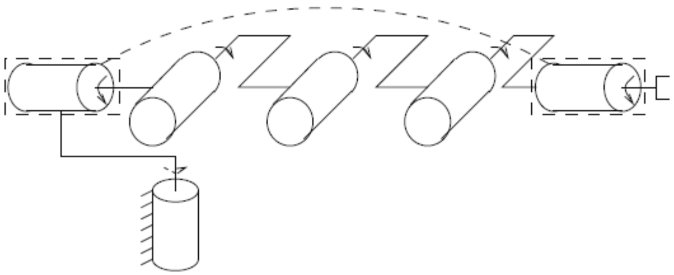
Singularities for 6R Manipulators

Case 1: Two Collinear Revolute Joints

$J(\theta)$ is singular if there exists two joints

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix}, \xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix}$$



s.t.

1. The axes are parallel, $\omega_1 = \pm\omega_2$
2. The axes are collinear, $\omega_i \times (q_1 - q_2) = 0, i = 1, 2$

- Elementary row or column operation do not change rank of $J(\theta)$:

$$J(\theta) = \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times q_2 & \cdots \\ \omega_1 & \omega_2 & \cdots \end{bmatrix} \in \mathbb{R}^{6\times n} \xrightarrow{\omega_1=\omega_2}$$

$$J(\theta) \sim \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times (q_2 - q_1) & \cdots \\ \omega_1 & 0 & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} -\omega_1 \times q_1 & 0 & \cdots \\ \omega_1 & 0 & \cdots \end{bmatrix}$$

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Singularities for 6R Manipulators

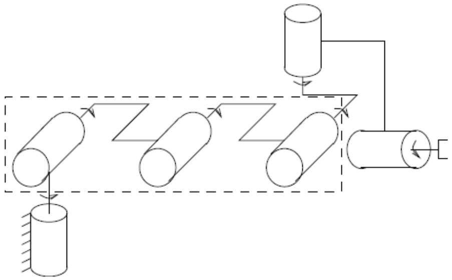
Case 2: Three Parallel Coplanar Revolute Joint Axes

$J(\theta)$ is singular if there exists three joints s.t.

1. The axes are parallel, $\omega_i = \pm\omega_j, i, j = 1, 2, 3$
2. The axes are coplanar, i.e. there exists a plane with normal n s.t.

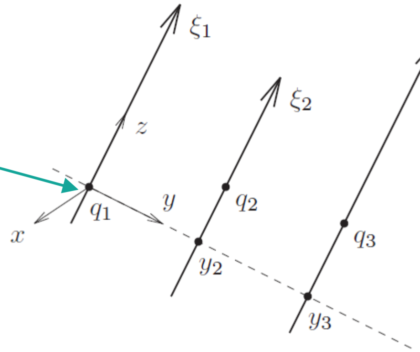
$$n^T\omega_i = 0, n^T(q_i - q_j) = 0, i, j = 1, 2, 3$$

$$J(\theta) = \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times (q_2 - q_1) & \cdots \\ \omega_1 & \omega_2 & \cdots \end{bmatrix}$$



$$\mathrm{Ad}_g J(\theta) = \underbrace{\begin{bmatrix} 0 & \pm y_2 & \pm y_3 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 1 & \pm 1 & \pm 1 & \cdots \end{bmatrix}}_{\text{Linearly dependent}}$$

adjoint transformation



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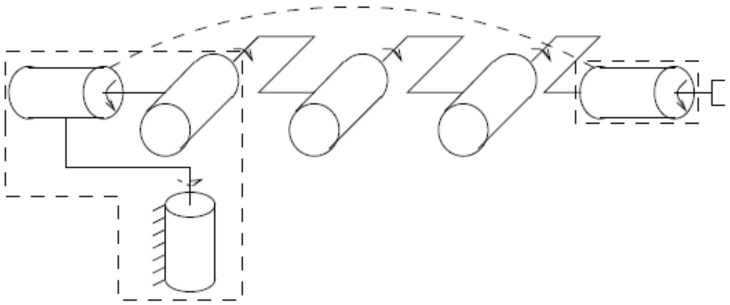
Singularities for 6R Manipulators

Case 3: Four Intersecting Revolute Joints Axes

$J(\theta)$ is singular if there exists four concurrent revolute joints with intersection point q s.t.:

$$\omega_i \times (q_i - q) = 0, i = 1, \ldots, 4$$

- Choose the frame origin at *q*,



$$p = q_i, i = 1, \ldots, 4$$

$$J(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 & \cdots \end{bmatrix}$$

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Manipulability

1. The ability to reach a certain position or set of positions => Workspace (complete/reachable/dextrous)

2. The ability to change the position or orientation at a given configuration => around a given local configuration

- **Jacobian relation of $g : \theta \in Q \mapsto g(\theta) \in SE(3)$**

$$V = J(\theta)\dot{\theta} \qquad (*)$$

- **Inverse Jacobian:**

Given $v \in \mathbb{R}^n$, solve for $\dot{\theta} \in \mathbb{R}^n$ from $(*)$

- **Application: Kinematic control by Inverse Jacobian**
  - Input: A desired $g_d(t) \in SE(3), t \in [0, T]$
  - Output: $\theta(k) = \theta(k\Delta T), \Delta T:$ Sampling period, $k = 1, \ldots, N = [T/\Delta T]$
  - Step 1: Let $g_d(k+1) = g(k)e^{\hat{V}\Delta T} = g(\theta(k))e^{\hat{V}\Delta T}$, solve for

$$\hat{V}\Delta T = \log(g^{-1}(k) \cdot g_d(k+1))$$

  - Step 2: Solve for $\dot{\theta}(k)$ from $V = J(\theta(k)) \cdot \dot{\theta}(k)$ and update

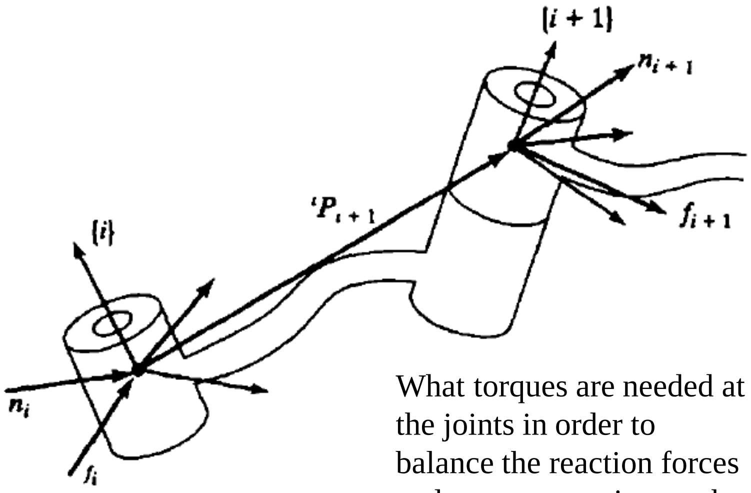
$$\theta(k+1) = \theta(k) + \dot{\theta}(k)\Delta T$$

Static Forces In Manipulators

How forces and moments "propagate" from one link to the next

- Solve for the **set of joint torques** needed to support a static load acting **at the end-effector**
  1. **Lock all the joints** so that the manipulator becomes a structure.
  2. Consider each link in this structure and **write a force-moment balance relationship** in terms of each link frames
  3. **Compute what static torque** must be acting *about the joint axis* in order for the manipulator to be in static equilibrium

f_i = force exerted on link i by link $i-1$,
n_i = torque exerted on link i by link $i-1$.



What torques are needed at the joints in order to balance the reaction forces and moments acting on the links?

$$ {}^if_i = {}^if_{i+1}, $$
$$ {}^in_i = {}^in_{i+1} + {}^iP_{i+1} \times {}^if_{i+1}. $$

$$ {}^if_i = {}^i_{i+1}R\, {}^{i+1}f_{i+1}, $$
$$ {}^in_i = {}^i_{i+1}R\, {}^{i+1}n_{i+1} + {}^iP_{i+1} \times {}^if_i. $$

$$ \tau_i = {}^in_i^T\, {}^i\hat{Z}_i $$ Revolute joints

$$ \tau_i = {}^if_i^T\, {}^i\hat{Z}_i $$ Prismatic joints

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A two-link manipulator is applying a force vector ${}^3F$ with its end-effector. (Consider this force to be acting at the origin of {3}.) Find the required joint torques as a function of configuration and of the applied force.

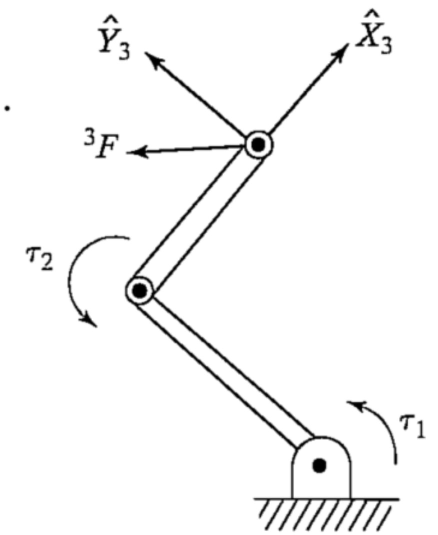
Example

1. *Lock all joints for a structure*
2. *Write Force equilibrium (From End to Base)*
3. *Compute Static Torque about the joint axis*

$${}^i f_i = {}^i_{i+1}R\, {}^{i+1} f_{i+1},$$

$${}^i n_i = {}^i_{i+1}R\, {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_i.$$

$$\tau_i = {}^i n_i^T \, {}^i \hat{Z}_i$$



$${}^2 f_2 = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix},$$

$${}^2 n_2 = l_2 \hat{X}_2 \times \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix},$$

$${}^1 f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 f_x - s_2 f_y \\ s_2 f_x + c_2 f_y \\ 0 \end{bmatrix},$$

$${}^1 n_1 = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} + l_1 \hat{X}_1 \times {}^1 f_1 = \begin{bmatrix} 0 \\ 0 \\ l_1 s_2 f_x + l_1 c_2 f_y + l_2 f_y \end{bmatrix}.$$

$$\tau_1 = l_1 s_2 f_x + (l_2 + l_1 c_2) f_y,$$

$$\tau_2 = l_2 f_y.$$

$$\tau = \begin{bmatrix} l_1 s_2 & l_2 + l_1 c_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

the transpose of the Jacobian

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Jacobians In The Force Domain

Use **Virtual Work** principal to analyze the static case by allowing **infinitesimal displacement**

6 x 1 Cartesian force-moment vector acting at the end-effector

6 x 1 vector of torques at the joints

When the Jacobian loses full rank, there are certain directions in which the end effector cannot exert static forces even if desired.

$$\mathcal{F} \cdot \delta\chi = \tau \cdot \delta\Theta \rightarrow \mathcal{F}^T \delta\chi = \tau^T \delta\Theta$$

6 x 1 infinitesimal Cartesian displacement of the end-effector

6 x 1 vector of infinitesimal joint displacements

$$\delta\chi = J\delta\Theta$$

Jacobian Definition

$$\mathcal{F}^T J\delta\theta = \tau^T \delta\Theta$$

Must hold true for all infinitesimal displacements

$$\mathcal{F}^T J = \tau^T$$

$$\tau = J^T \mathcal{F}$$

Allows us to convert a Cartesian quantity into a joint-space quantity without calculating any inverse kinematic functions

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Instantaneous Cartesian Transformation

Velocities & Static Forces

- 6x1 general velocity of a body $\nu = \begin{bmatrix} v \\ \omega \end{bmatrix}$
- 6x1 general force vector $\mathcal{F} = \begin{bmatrix} F \\ N \end{bmatrix}$

$$P\times = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1})$$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

Set to zero here assuming frames A & B are rigidly connected.

$$\begin{bmatrix} ^{B}v_B \\ ^{B}\omega_B \end{bmatrix} = \begin{bmatrix} ^{B}_{A}R & -^{B}_{A}R\,{}^{A}P_{BORG}\times \\ 0 & ^{B}_{A}R \end{bmatrix} \begin{bmatrix} ^{A}v_A \\ ^{A}\omega_A \end{bmatrix}$$

$$^{B}\nu_B = {}^{B}_{A}T_v\,{}^{A}\nu_A$$

$$^{A}\nu_A = {}^{A}_{B}T_v\,{}^{B}\nu_B$$

Velocity Transformation T_v

$$^{A}\mathcal{F}_A = {}^{A}_{B}T_f\,{}^{B}\mathcal{F}_B$$

Force-moment Transformation T_f

$$\begin{bmatrix} ^{A}F_A \\ ^{A}N_A \end{bmatrix} = \begin{bmatrix} ^{A}_{B}R & 0 \\ ^{A}P_{BORG}\times {}^{A}_{B}R & ^{A}_{B}R \end{bmatrix} \begin{bmatrix} ^{B}F_B \\ ^{B}N_B \end{bmatrix}$$

$$\begin{bmatrix} ^{A}v_A \\ ^{A}\omega_A \end{bmatrix} = \begin{bmatrix} ^{A}_{B}R & ^{A}P_{BORG}\times {}^{A}_{B}R \\ 0 & ^{A}_{B}R \end{bmatrix} \begin{bmatrix} ^{B}v_B \\ ^{B}\omega_B \end{bmatrix}$$

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Thank you!

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