Lecture 02
Mathematical Foundations

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A Future of Human-Robot Competition?

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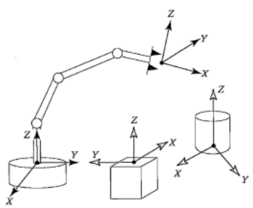
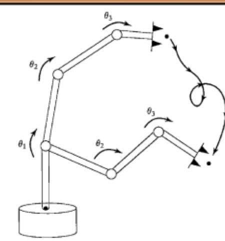
Mechanics & Control

A Typical Knowledge Flow of Mechanical Manipulators

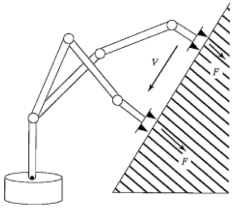
Description of Position and Orientation



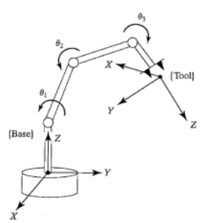
Linear & Nonlinear Position Control



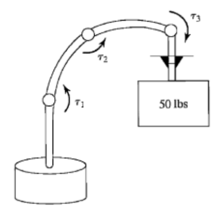
Force Control



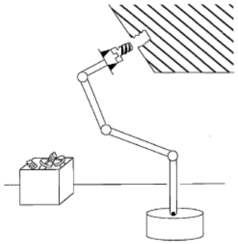
Forward Kinematics of Manipulators



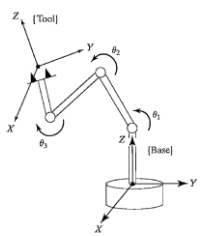
Manipulator Design and Sensors



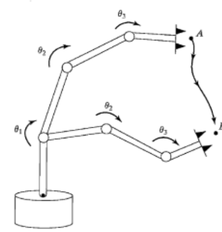
Programming Robots



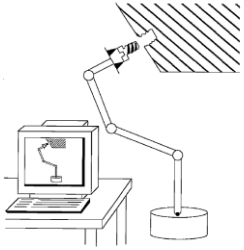
Inverse Kinematics of Manipulators



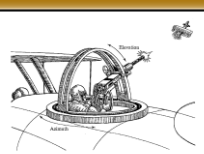
Trajectory Generation



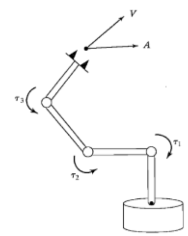
Off-line Programming and Simulation

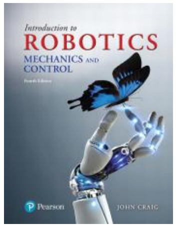


Velocities, Static Forces, Singularities



Dynamics





Introduction to Robotics: Mechanics and Control

By John Craig

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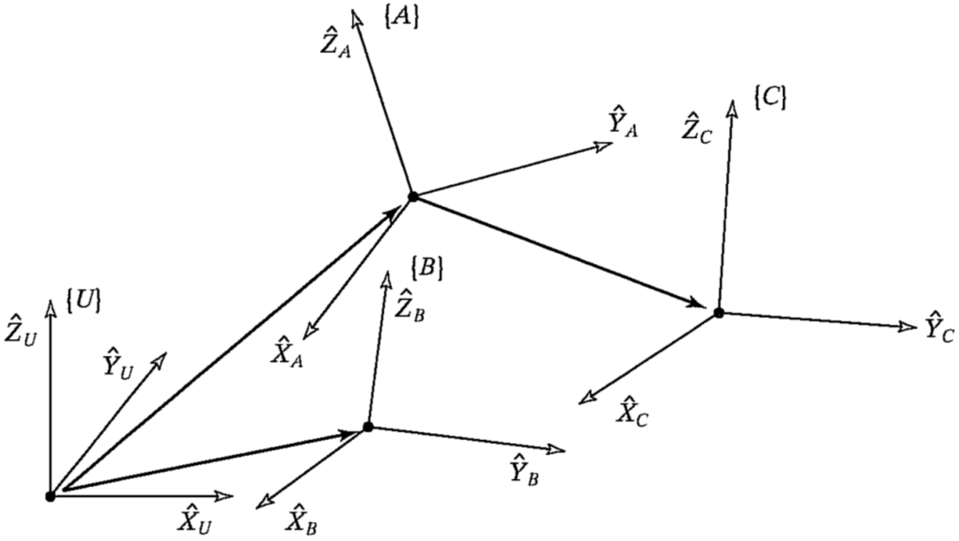
A Universe Coordinate System

The assumption that everything is referenced to …

- A coordinate system attached to the World Frame

- All positions and orientations w. r. t.
  - The Universe Coordinate System, or
  - Other Cartesian CS that are (or could be) defined relative to the Universe CS.

- **Robotic Mechanisms** are systems of rigid bodies connected by joints.
- **Pose** is the collective term of the *position* and *orientation* of a rigid body in space.



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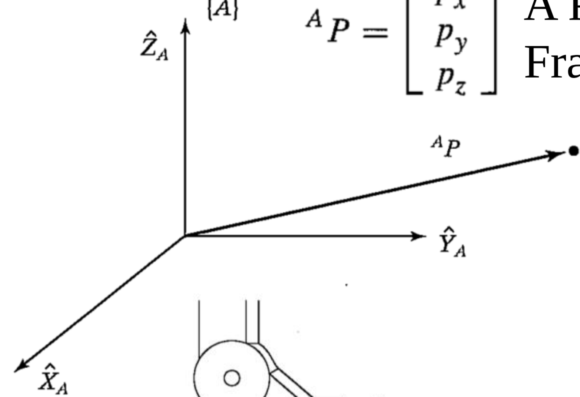
Position & Orientation

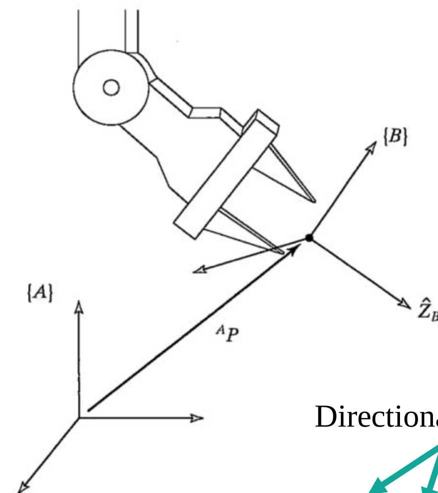
For $p \in \mathbb{R}^n$, $n = 2$ for planar, $n = 3$ for spatial

- Point: $p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$, $\|p\| = \sqrt{p_1^2 + \cdots + p_n^2}$



$${}^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$ A Point in Frame {A}

- Vector: $v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ \vdots \\ p_n - q_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$



Rotation of Frame {B} relative to Frame {A}

- Matrix: $A \in \mathbb{R}^{n \times m}$, $A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$

Directional cosines

$${}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$${}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T}$$

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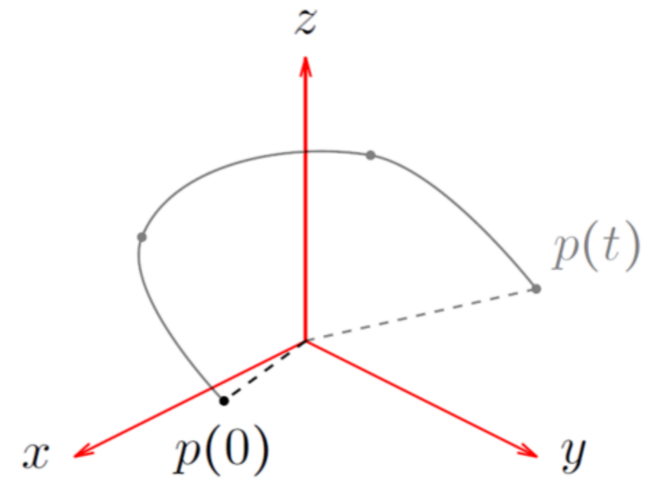
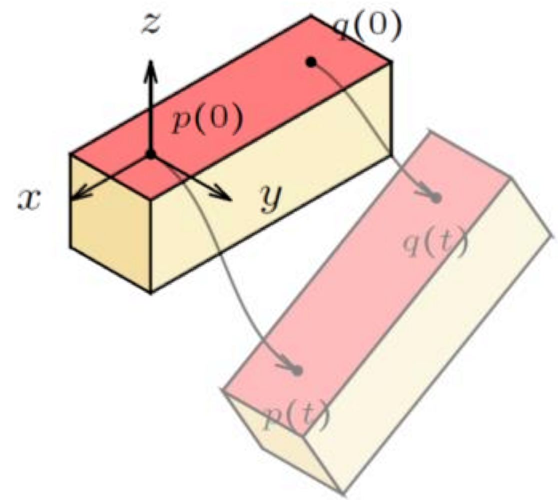
Description of Point-Mass Motion

Rigid-body Assumption

- $p(0) = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}$: initial position
- $p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, t \in (-\epsilon, \epsilon)$



- Trajectory
  - A curve $p: (-\epsilon, \epsilon) \longmapsto \mathbb{R}^3, p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$
- Rigid body transformation
  - $\|p(t) - q(t)\| = \|p(0) - q(0)\| = constant$



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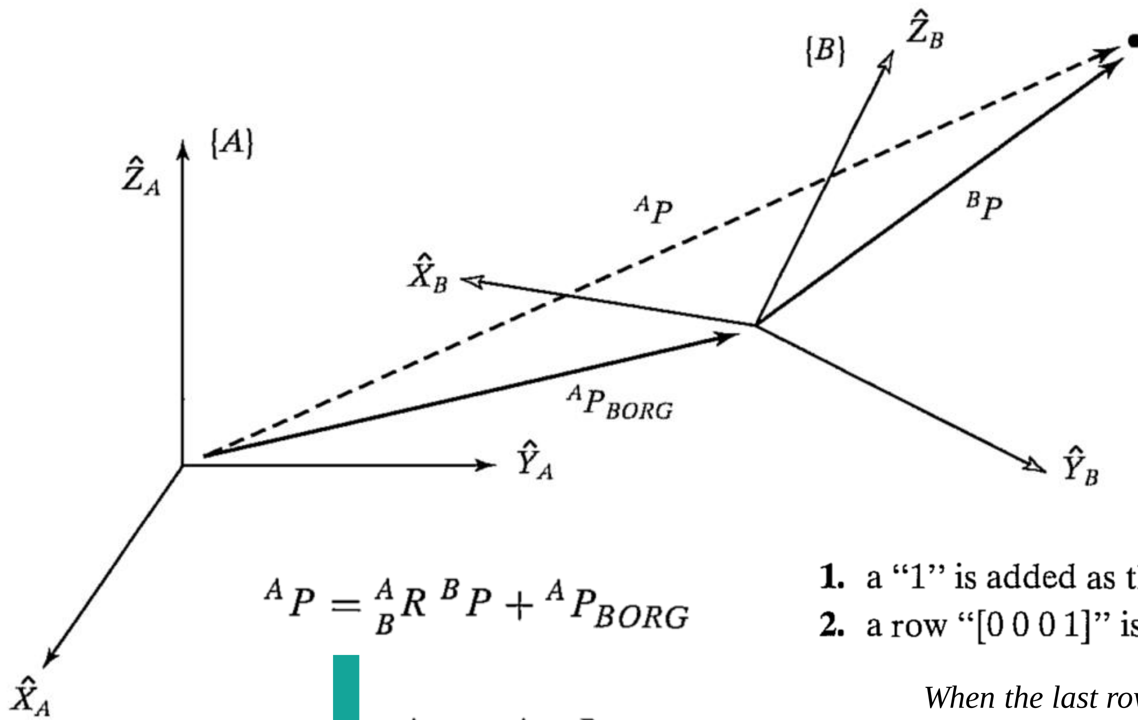
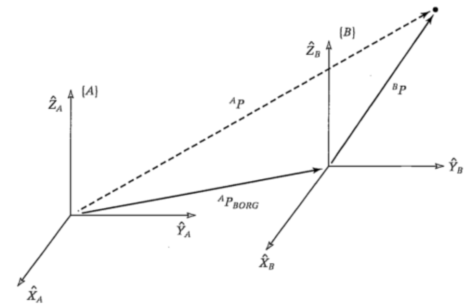
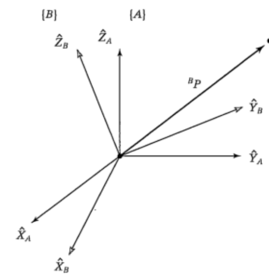
Homogeneous Transformation

Translation + Rotation = Transformation







$${}^{A}P = {}^{A}_{B}R\, {}^{B}P + {}^{A}P_{BORG}$$

$${}^{A}P = {}^{A}_{B}T\, {}^{B}P$$

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$$

$${}^{A}P = {}^{A}_{B}R\, {}^{B}P + {}^{A}P_{BORG}$$
$$1 = 1.$$

1. a "1" is added as the last element of the 4×1 vectors;
2. a row "[0 0 0 1]" is added as the last row of the 4×4 matrix.

When the last row is other than [0 0 0 1] or the rotation matrix is not orthonormal, this 4x4 matrix can be also used to compute perspective and scaling operations

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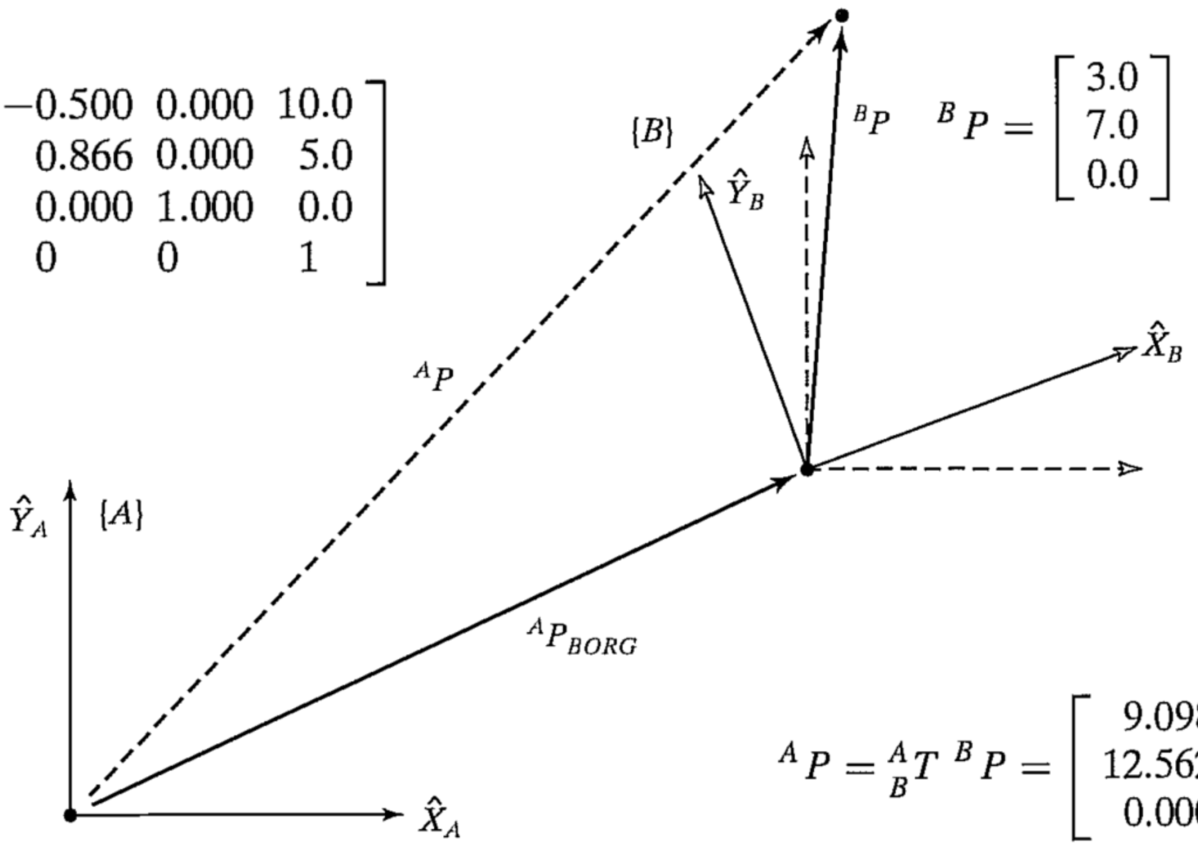
Exercise

Figure 2.8 shows a frame $\{B\}$, which is rotated relative to frame $\{A\}$ about $\hat{Z}$ by 30 degrees, translated 10 units in $\hat{X}_A$, and translated 5 units in $\hat{Y}_A$. Find ${}^{A}P$, where ${}^{B}P = [3.07.00.0]^T$.

$$
{}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



$$
{}^{B}P = \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \end{bmatrix}
$$

$$
{}^{A}P = {}^{A}_{B}T\, {}^{B}P = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix}
$$

Interpretations of Transformation

Three ways

- **It is a description of a frame.**
  - ${}^{A}_{B}T$ describes the frame {B} relative to the frame {A}. Specifically, the columns of ${}^{A}_{B}R$ are unit vectors defining the directions of the principal axes of {B}, and ${}^{A}P_{BORG}$ locates the position of the origin of {B}.

- **It is a transform mapping.**
  - ${}^{A}_{B}T$ maps ${}^{B}P \rightarrow {}^{A}P$

- **It is a transform operator.**
  - T operates on ${}^{A}P_1$ to create ${}^{A}P_2$

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Common Operators

Translational / Rotational / Transformation

- Translational Operator

$$^{A}P_2 = D_Q(q)\,^{A}P_1$$

$$\mathrm{Trans}(a,b,c) = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Rotational Operator

$$^{A}P_2 = R_K(\theta)\,^{A}P_1$$

$$\mathrm{Rot}_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{Rot}_y(\theta_y) = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{Rot}_z(\theta_z) = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Transformation Operator

$$^{A}P_2 = T\,^{A}P_1$$

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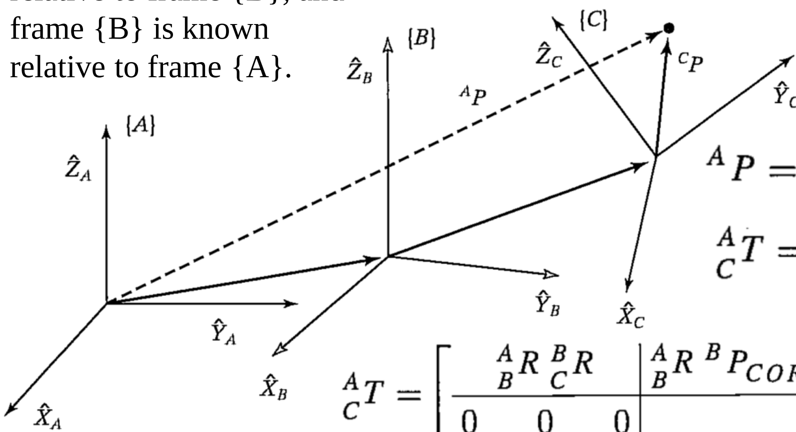
Transformation Arithmetic

Compound / Inversion / Equation

Frame {C} is known relative to frame {B}, and frame {B} is known relative to frame {A}.



$$ {}^{A}P = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P $$

$$ {}^{A}_{C}T = {}^{A}_{B}T\,{}^{B}_{C}T $$

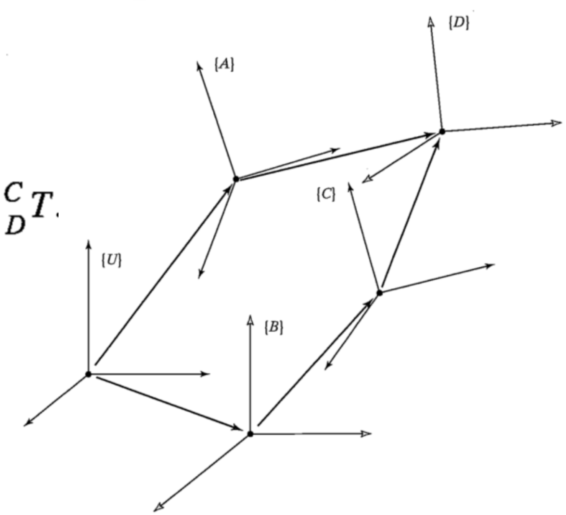
$$ {}^{A}_{C}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R\,{}^{B}_{C}R & & {}^{A}_{B}R\,{}^{B}P_{CORG} + {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Avoid direct inverse operation

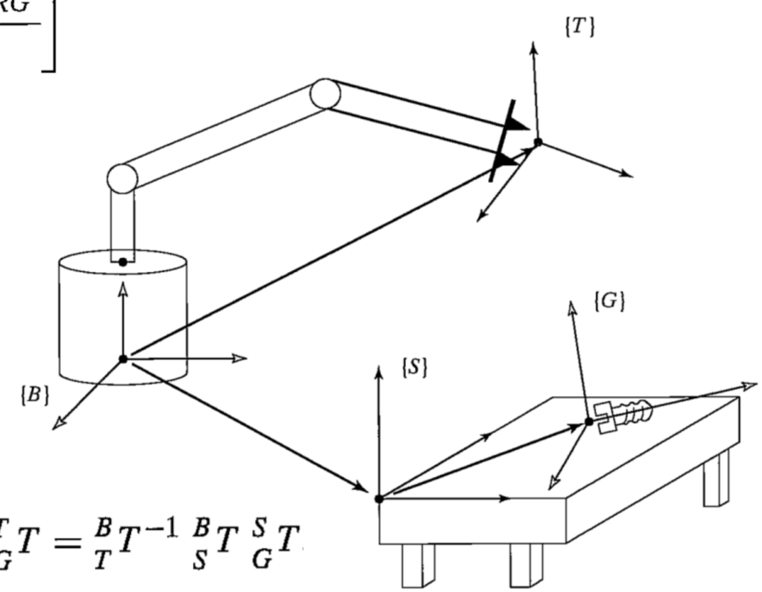
- Computationally expensive in practice

A general and extremely useful way of computing the inverse of a homogeneous transform.

$$ {}^{B}_{A}T = {}^{A}_{B}T^{-1} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$



$$ {}^{U}_{A}T\,{}^{A}_{D}T = {}^{U}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T. $$



$$ {}^{T}_{G}T = {}^{B}_{T}T^{-1}\,{}^{B}_{S}T\,{}^{S}_{G}T $$

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Proper Orthonormal Matrix

Det = +1 Orthogonal + Normalized

- Cayley's Formula for orthonormal matrices

a skew-symmetric matrix

Any proper orthonormal matrix

$$R = (I_3 - S)^{-1}(I_3 + S)$$

$$S = \begin{bmatrix} 0 & -s_x & s_y \\ s_x & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}$$

  - *Any 3 x 3 rotation matrix can be specified by just 3 parameters*

- But rotations don't usually commute

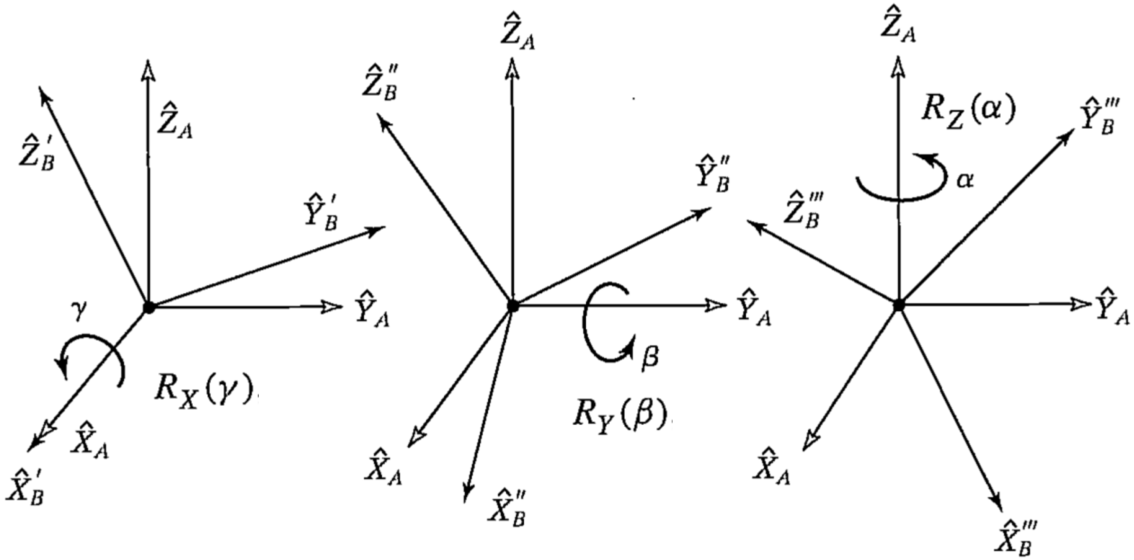
$$R_z(30) = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \qquad R_z(30)R_x(30) = \begin{bmatrix} 0.87 & -0.43 & 0.25 \\ 0.50 & 0.75 & -0.43 \\ 0.00 & 0.50 & 0.87 \end{bmatrix}$$

$$R_x(30) = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.866 & -0.500 \\ 0.000 & 0.500 & 0.866 \end{bmatrix} \qquad \neq R_x(30)R_z(30) = \begin{bmatrix} 0.87 & -0.50 & 0.00 \\ 0.43 & 0.75 & -0.50 \\ 0.25 & 0.43 & 0.87 \end{bmatrix}$$

- ***<u>How to construct a simpler representation with the minimal (three) numbers?</u>***

X-Y-Z Fixed Angles

Roll-Pitch-Yaw Angles



$${}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma)$$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$$= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

$${}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}),$$

$$\alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta),$$

$$\gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta),$$

Atan2(y, x) is a two-argument arc tangent function

- Using the positive square root, we can obtain single solution between $[-\pi, \pi]$
- If $\beta = \pm 90.0°$, then we can choose the following cases.

$\beta = 90.0°$, $\alpha = 0.0$, $\gamma = \text{Atan2}(r_{12}, r_{22})$.

or

$\beta = -90.0°$, $\alpha = 0.0$, $\gamma = -\text{Atan2}(r_{12}, r_{22})$.

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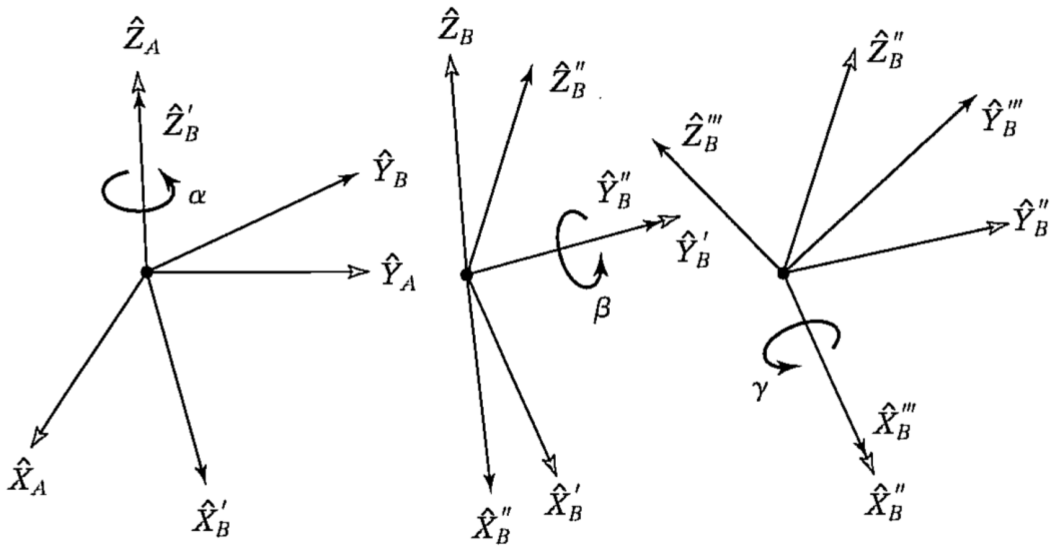
Z-Y-X Euler Angles

W.R.T. the Moving System {B} instead of the Fixed {A}



$${}^A_B R = {}^A_{B'}R \; {}^{B'}_{B''}R \; {}^{B''}_{B}R$$

Three rotations taken about fixed axes yield the same final orientation as the same three rotations taken in opposite order about the axes of the moving frame.

$${}^A_B R_{Z'Y'X'} = R_Z(\alpha) R_Y(\beta) R_X(\gamma)$$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$${}^A_B R_{Z'Y'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

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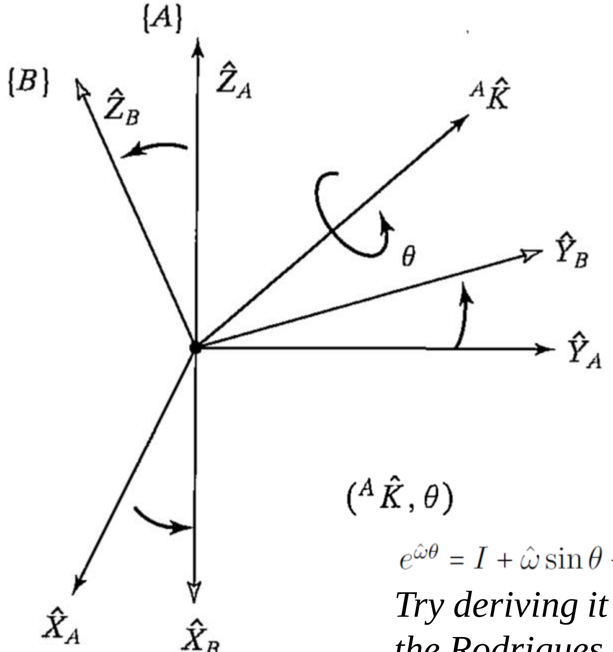
Equivalent Angle-Axis Representation

If the axis is a general direction (rather than one of the unit directions), any orientation may be obtained through proper axis and angle selection

Start with the frame coincident with a known frame $\{A\}$; then rotate $\{B\}$ about the vector ${}^{A}\hat{K}$ by an angle θ according to the right-hand rule.



$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$

Try deriving it yourself using the Rodrigues equation

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

where $c\theta = \cos\theta$, $s\theta = \sin\theta$, $v\theta = 1 - \cos\theta$, and ${}^{A}\hat{K} = [k_x k_y k_z]^T$

$${}^{A}_{B}R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\theta = \mathrm{Acos}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

- Simple form of representation
- θ falls between 0 and π
- Two solutions problem
- Becomes ill-defined for small angular rotations

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Euler Parameters as a Unit Quaternion

Evolving from Equivalent Angle-Axis to A Four-Parameter System

- If equivalent axis $\hat{K} = [k_x \; k_y \; k_z]^T$ with equivalent angle θ

$$\epsilon_1 = k_x \sin\frac{\theta}{2},$$
$$\epsilon_2 = k_y \sin\frac{\theta}{2},$$
$$\epsilon_3 = k_z \sin\frac{\theta}{2},$$
$$\epsilon_4 = \cos\frac{\theta}{2}.$$

$$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$$

A Unit Quaternion as a 4x1 vector

Rotation matrix written in Euler Parameters

$$R_\epsilon = \begin{bmatrix} 1-2\epsilon_2^2-2\epsilon_3^2 & 2(\epsilon_1\epsilon_2-\epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3+\epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2+\epsilon_3\epsilon_4) & 1-2\epsilon_1^2-2\epsilon_3^2 & 2(\epsilon_2\epsilon_3-\epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3-\epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3+\epsilon_1\epsilon_4) & 1-2\epsilon_1^2-2\epsilon_2^2 \end{bmatrix}$$

$$\epsilon_1 = \frac{r_{32}-r_{23}}{4\epsilon_4},$$
$$\epsilon_2 = \frac{r_{13}-r_{31}}{4\epsilon_4},$$
$$\epsilon_3 = \frac{r_{21}-r_{12}}{4\epsilon_4},$$
$$\epsilon_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}.$$

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Computation Considerations

Practical Reality

- Homogeneous representation requires wasteful time multiplying by zeros and ones
  - The availability of inexpensive computing power is largely responsible for the growth of the robotics industry;
  - yet, for some time to come, efficient computation will remain an important issue in the design of a manipulation system.
- Order of multiplication
  - ${}^{A}P = {}^{A}_{D}R\,{}^{D}P$ 63 multiplications and 42 additions
  - ${}^{A}P = {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}_{D}R\,{}^{D}P$ 27 multiplications and 18 additions

$${}^{A}P = {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}P$$

$${}^{A}P = {}^{A}_{B}R\,{}^{B}P$$

$${}^{A}P = {}^{A}P,$$

$${}^{A}P = {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}_{D}R\,{}^{D}P.$$

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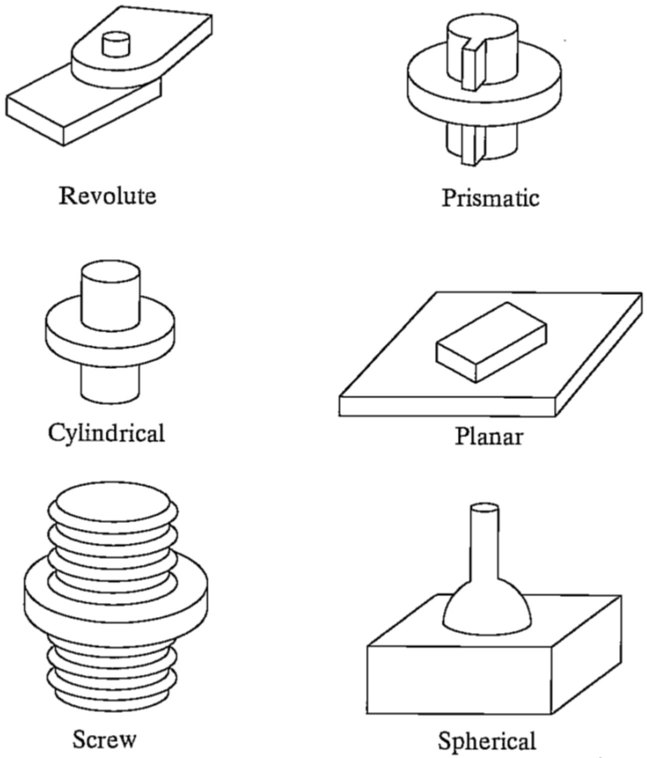
Kinematics

The science of motion that treats the subject without regard to the forces that cause it

- A ***manipulator*** may be thought of as a set of bodies connected in a chain by joints
  - These bodies are called *links*.
  - *Joints* form a connection between a neighboring pair of links
    - *Lower-pair Joints*: two surfaces sliding over one another
    - Most manipulators have **revolute** or **prismatic** joints, both having 1 degree of freedom



- Manipulator Kinematics
  - All the geometrical and time-based properties of the motion
    - *The position, the velocity, the acceleration, and all higher order derivatives of the position variables (w.r.t. time or any other variable(s))*

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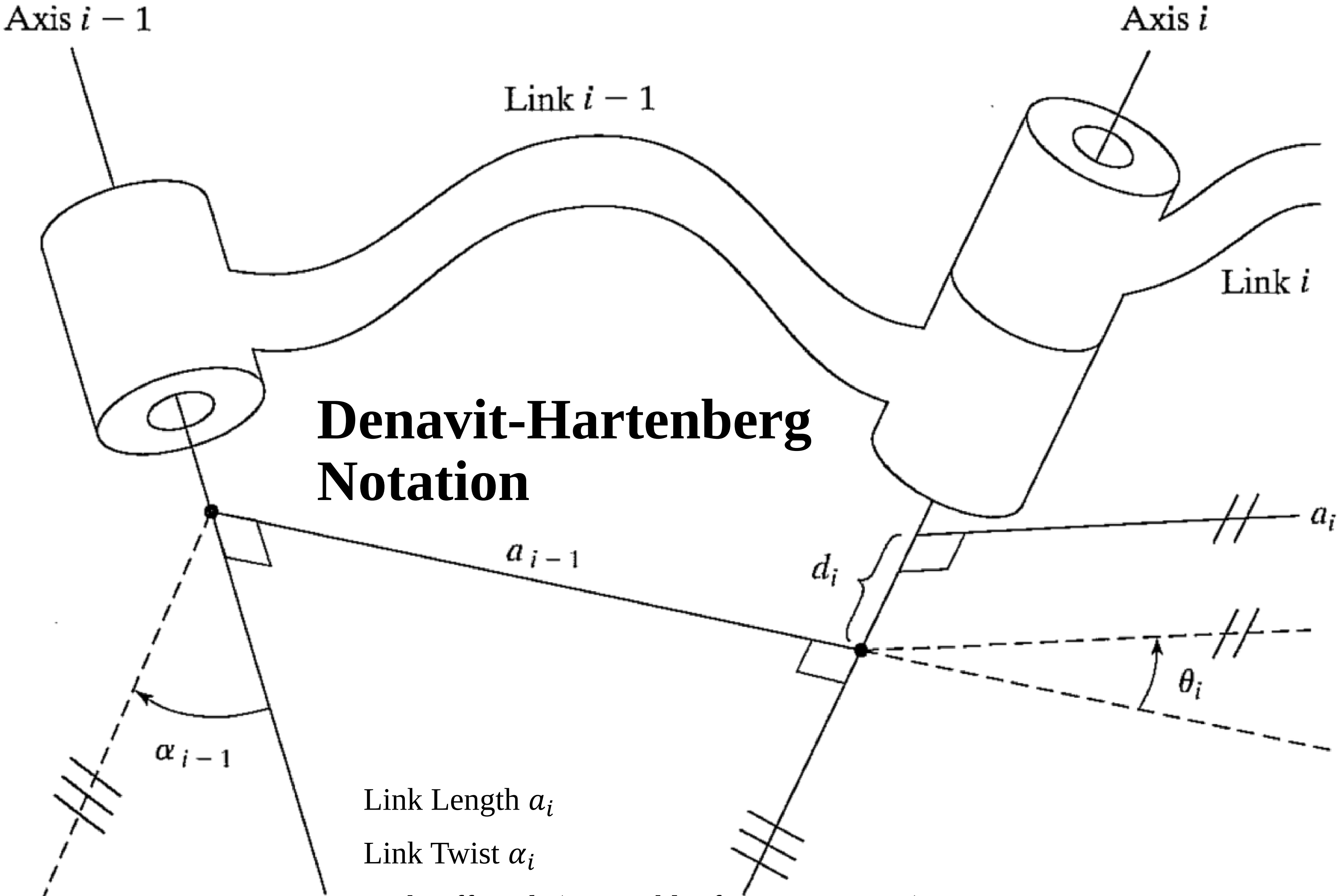

Denavit-Hartenberg Notation

Link Length a_i

Link Twist α_i

Link Offset d_i (A variable if prismatic joint)

Joint Angle θ_i (A variable if revolute joint)

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DH Parameters

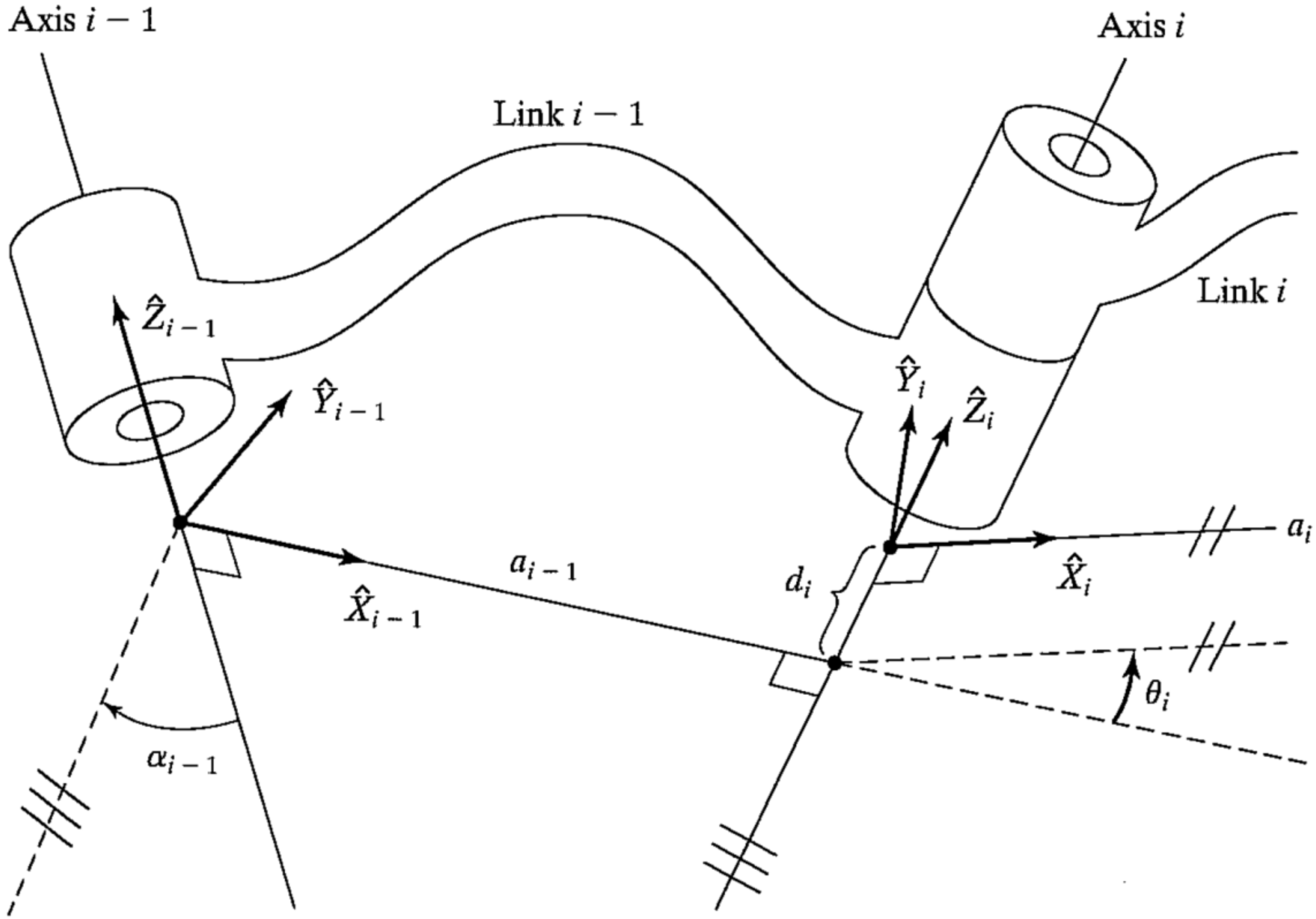
Frame Attachment

$a_i =$ *the distance from* $\hat{Z}_i$ *to* $\hat{Z}_{i+1}$ *measured along* $\hat{X}_i$;

$\alpha_i =$ *the angle from* $\hat{Z}_i$ *to* $\hat{Z}_{i+1}$ *measured about* $\hat{X}_i$;

$d_i =$ *the distance from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured along* $\hat{Z}_i$; and

$\theta_i =$ *the angle from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured about* $\hat{Z}_i$.



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Uniqueness of Link Parameters

Practical Considerations

- Two choices of joint axis direction, $\widehat{Z}_i$
- Two choices of link length axis direction, $\widehat{X}_i$, for intersecting joints (i.e. $a_i = 0$)
- Arbitrary choice of origin when axes are parallel
- Freedom in frame assignment with prismatic joints

- Meaning that there could be multiple ways of writing the DH parameters, depending on the different choice of frame assignment.
  - Also multiple ways of interpreting the calculated results, if using different ways of frame assignment

- *Careful, or you might get lost*

Procedure of Link-Frame Attachment

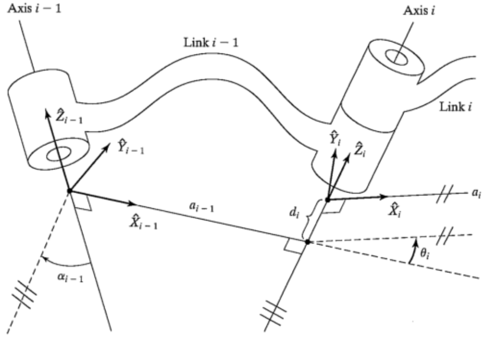
1. Identify the joint axes and imagine (or draw) infinite lines along them. For steps 2 through 5 below, consider two of these neighboring lines (at axes i and $i+1$).
2. Identify the common perpendicular between them, or point of intersection. At the point of intersection, or at the point where the common perpendicular meets the ith axis, assign the link-frame origin.
3. Assign the $\hat{Z}_i$ axis pointing along the ith joint axis.
4. Assign the $\hat{X}_i$ axis pointing along the common perpendicular, or, if the axes intersect, assign $\hat{X}_i$ to be normal to the plane containing the two axes.
5. Assign the $\hat{Y}_i$ axis to complete a right-hand coordinate system.
6. Assign {0} to match {1} when the first joint variable is zero. For $\{N\}$, choose an origin location and $\hat{X}_N$ direction freely, but generally so as to cause as many linkage parameters as possible to become zero.

$a_i =$ *the distance from* $\hat{Z}_i$ *to* $\hat{Z}_{i+1}$ *measured along* $\hat{X}_i$;

$\alpha_i =$ *the angle from* $\hat{Z}_i$ *to* $\hat{Z}_{i+1}$ *measured about* $\hat{X}_i$;

$d_i =$ *the distance from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured along* $\hat{Z}_i$; and

$\theta_i =$ *the angle from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured about* $\hat{Z}_i$.



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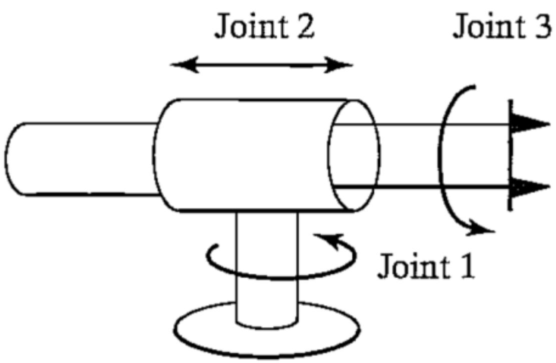
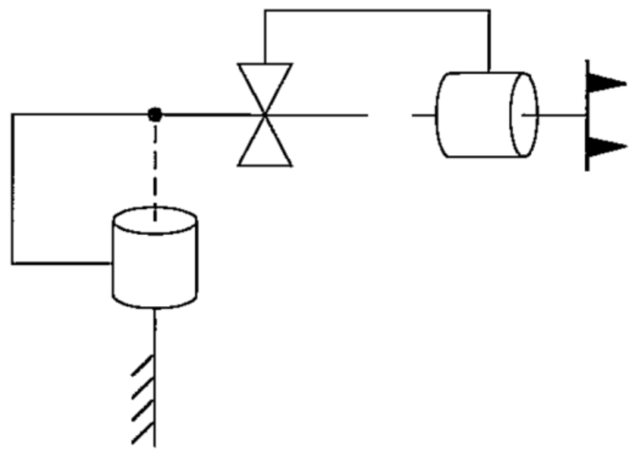
Exercise

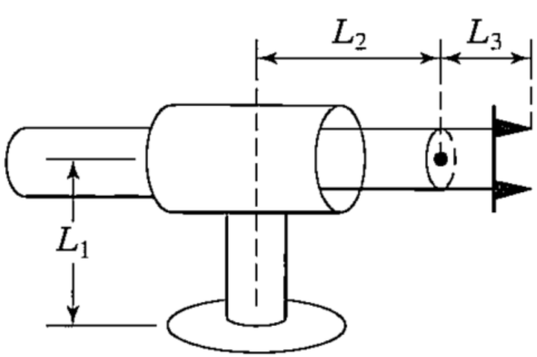
O => Z => X => a => α => d => θ

1. Identify the joint axes and imagine (or draw) infinite lines along them. For steps 2 through 5 below, consider two of these neighboring lines (at axes i and $i+1$).
2. Identify the common perpendicular between them, or point of intersection. At the point of intersection, or at the point where the common perpendicular meets the ith axis, assign the link-frame origin.
3. Assign the $\hat{Z}_i$ axis pointing along the ith joint axis.
4. Assign the $\hat{X}_i$ axis pointing along the common perpendicular, or, if the axes intersect, assign $\hat{X}_i$ to be normal to the plane containing the two axes.
5. Assign the $\hat{Y}_i$ axis to complete a right-hand coordinate system.
6. Assign {0} to match {1} when the first joint variable is zero. For $\{N\}$, choose an origin location and $\hat{X}_N$ direction freely, but generally so as to cause as many linkage parameters as possible to become zero.

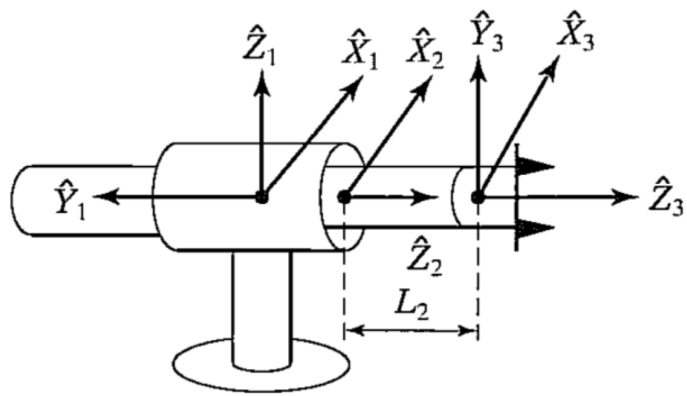




i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	90°	0	d_2	0
3	0	0	L_2	θ_3



(a)



(b)

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Derivation of Link Transformations

Each of the four transformations will be a function of one link parameter only and will be simple enough that we can write down its form by inspection

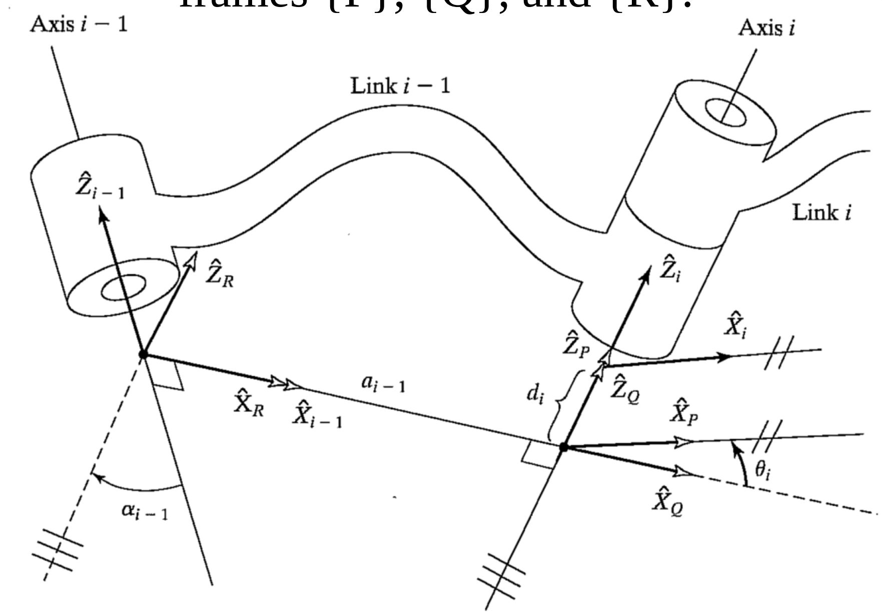
$$ {}^{i-1}_{i}T = {}^{i-1}_{R}T \; {}^{R}_{Q}T \; {}^{Q}_{P}T \; {}^{P}_{i}T $$

$$ {}^{i-1}_{i}T = R_X(\alpha_{i-1}) D_X(a_{i-1}) R_Z(\theta_i) D_Z(d_i) $$

$$ {}^{i-1}_{i}T = \mathrm{Screw}_X(a_{i-1}, \alpha_{i-1}) \; \mathrm{Screw}_Z(d_i, \theta_i) $$

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Location of intermediate frames {P}, {Q}, and {R}.



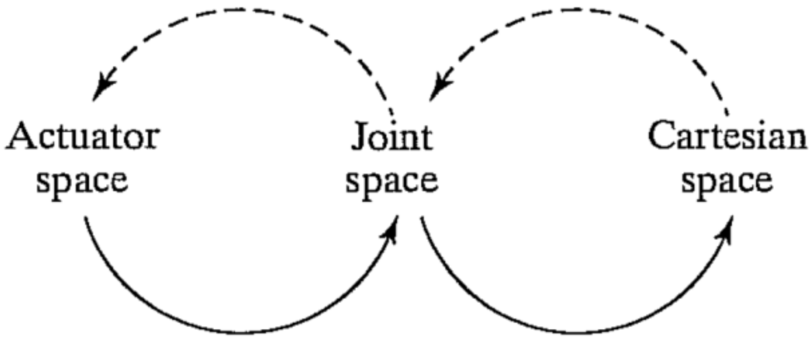
Concatenating link transformations

- The single transformation that relates frame {N} to frame {0}

$$ {}^{0}_{N}T = {}^{0}_{1}T \; {}^{1}_{2}T \; {}^{2}_{3}T \ldots {}^{N-1}_{N}T $$

a function of all n joint variables

Actuator Space, Joint Space, and Cartesian Space



- **Joint Space**
  - The space of all joint vectors.
  - A $n \times 1$ joint vector refers to a set of n joint variables specifying the position of all the links of a manipulator of n degrees of freedom.
- **Cartesian Space**
  - When position is measured along orthogonal axes and orientation is measured according to any Cartesian conventions.
- **Actuator Space**
  - The space of all actuator positions.
  - Computations necessary to realized the joint vector as a function of a set of actuator values.

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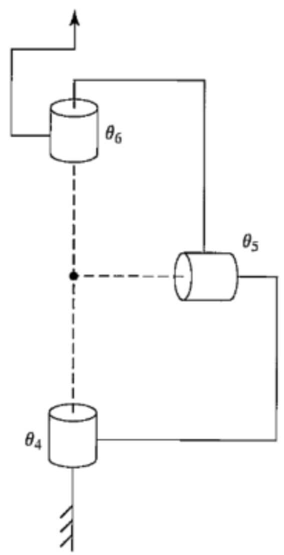
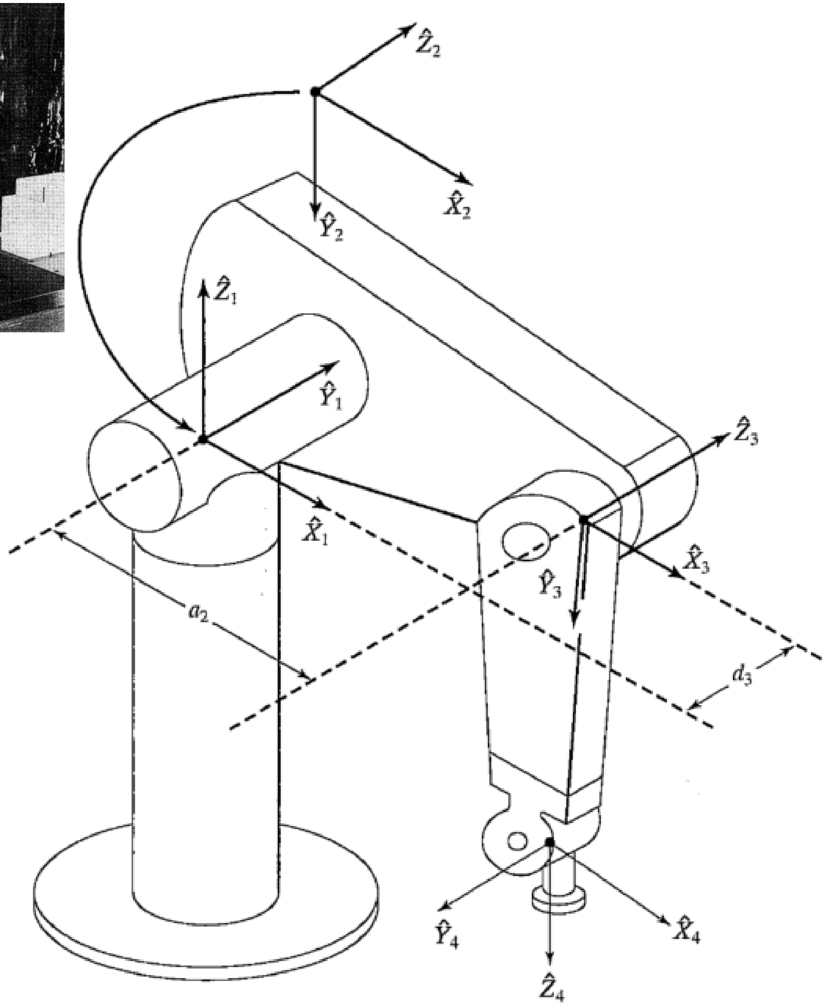
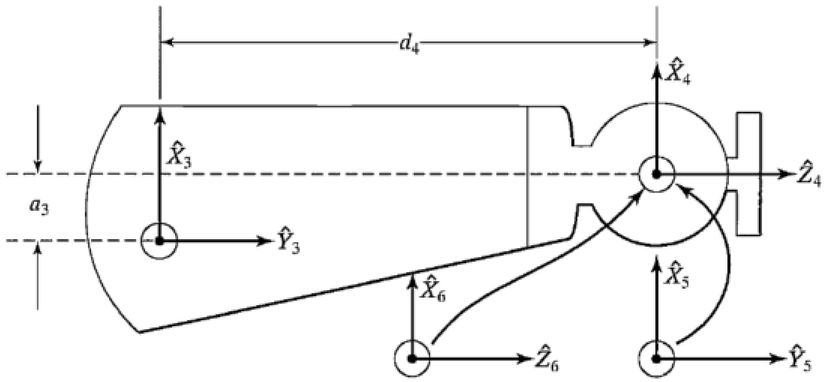
Example: Kinematics of PUMA 560

Attach the Frames => Determine the DH => Check & Revise

i	$\alpha_i - 1$	$a_i - 1$	d_i	θi
1	0	0	0	θ_1
2	−90°	0	0	θ_2
3	0	a_2	d_3	θ_3
4	−90°	a_3	d_4	θ_4
5	90°	0	0	θ_5
6	−90°	0	0	θ_6







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$$
{}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^2_3T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^3_4T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^4_5T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^5_6T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

$$
{}^4_6T = {}^4_5T\,{}^5_6T = \begin{bmatrix} c_5c_6 & -c_5s_6 & -s_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ s_5c_6 & -s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^3_6T = {}^3_4T\,{}^4_6T = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 & a_3 \\ s_5c_6 & -s_5s_6 & c_5 & d_4 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^1_3T = {}^1_2T\,{}^2_3T = \begin{bmatrix} c_{23} & -s_{23} & 0 & a_2c_2 \\ 0 & 0 & 1 & d_3 \\ -s_{23} & -c_{23} & 0 & -a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^1_6T = {}^1_3T\,{}^3_6T = \begin{bmatrix} {}^1r_{11} & {}^1r_{12} & {}^1r_{13} & {}^1p_x \\ {}^1r_{21} & {}^1r_{22} & {}^1r_{23} & {}^1p_y \\ {}^1r_{31} & {}^1r_{32} & {}^1r_{33} & {}^1p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
{}^1r_{11} &= c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6, \\
{}^1r_{21} &= -s_4c_5c_6 - c_4s_6, \\
{}^1r_{31} &= -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6, \\
{}^1r_{12} &= -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6, \\
{}^1r_{22} &= s_4c_5s_6 - c_4c_6, \\
{}^1r_{32} &= s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6, \\
{}^1r_{13} &= -c_{23}c_4s_5 - s_{23}c_5, \\
{}^1r_{23} &= s_4s_5, \\
{}^1r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\
{}^1p_x &= a_2c_2 + a_3c_{23} - d_4s_{23}, \\
{}^1p_y &= d_3, \\
{}^1p_x &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
$$

$$
{}^0_6T = {}^0_1T\,{}^1_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
r_{11} &= c_1[c_{23}(c_4c_5c_6 - s_4s_5) - s_{23}s_5c_5] + s_1(s_4c_5c_6 + c_4s_6), \\
r_{21} &= s_1[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6 - c_1(s_4c_5c_6 + c_4s_6), \\
r_{31} &= -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6,
\end{aligned}
$$

$$
\begin{aligned}
r_{12} &= c_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] + s_1(c_4c_6 - s_4c_5s_6), \\
r_{22} &= s_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] - c_1(c_4c_6 - s_4c_5s_6), \\
r_{32} &= -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6,
\end{aligned}
$$

$$
\begin{aligned}
r_{13} &= -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5, \\
r_{23} &= -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5, \\
r_{33} &= s_{23}c_4s_5 - c_{23}c_5,
\end{aligned}
\qquad
\begin{aligned}
p_x &= c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1, \\
p_y &= s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] + d_3c_1, \\
p_z &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
$$

i	$\alpha_i - 1$	$a_i - 1$	d_i	θi
1	0	0	0	θ_1
2	$-90°$	0	0	θ_2
3	0	a_2	d_3	θ_3
4	$-90°$	a_3	d_4	θ_4
5	$90°$	0	0	θ_5
6	$-90°$	0	0	θ_6

The basic equations for all kinematic analysis of this manipulator

Attach the frames wisely to ease the computational burden

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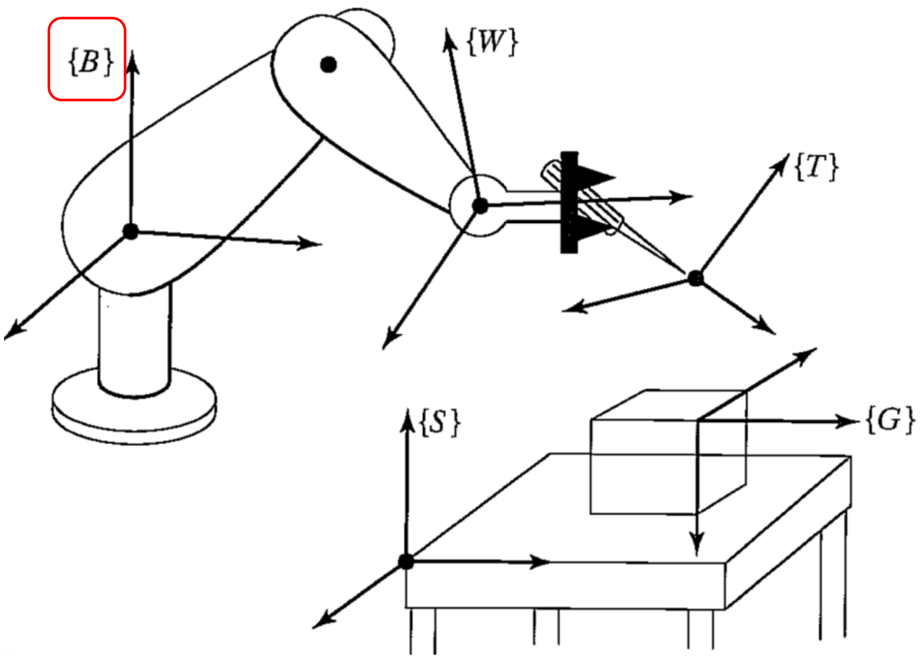
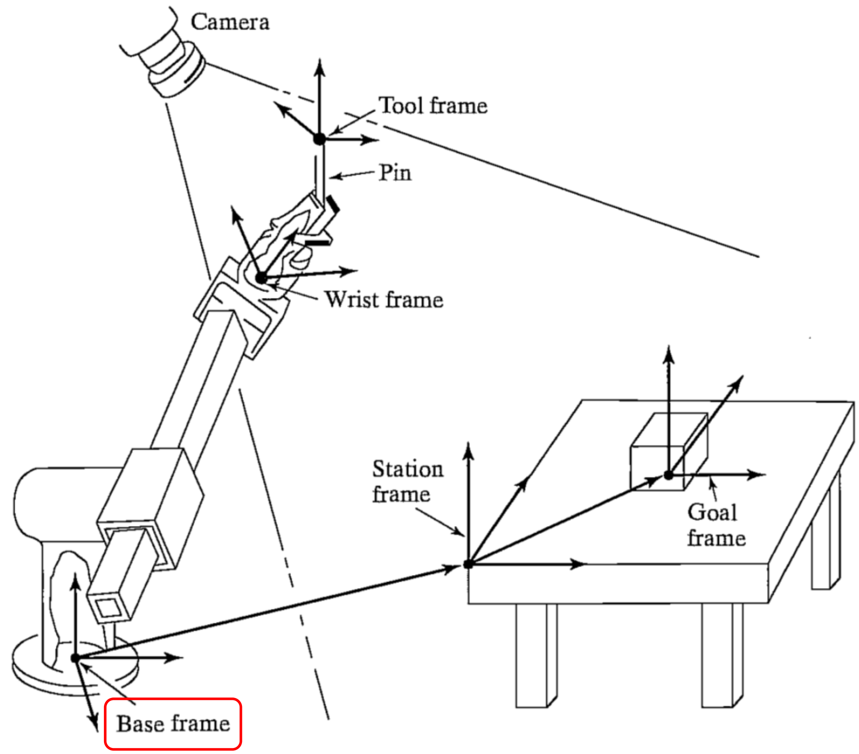
The Base Frame, {B}

Frames with Standard Names

- {B} is located at the base of the manipulator.
  - It is merely another name for frame {0}.
- It is affixed to a nonmoving part of the robot, sometimes called link 0.

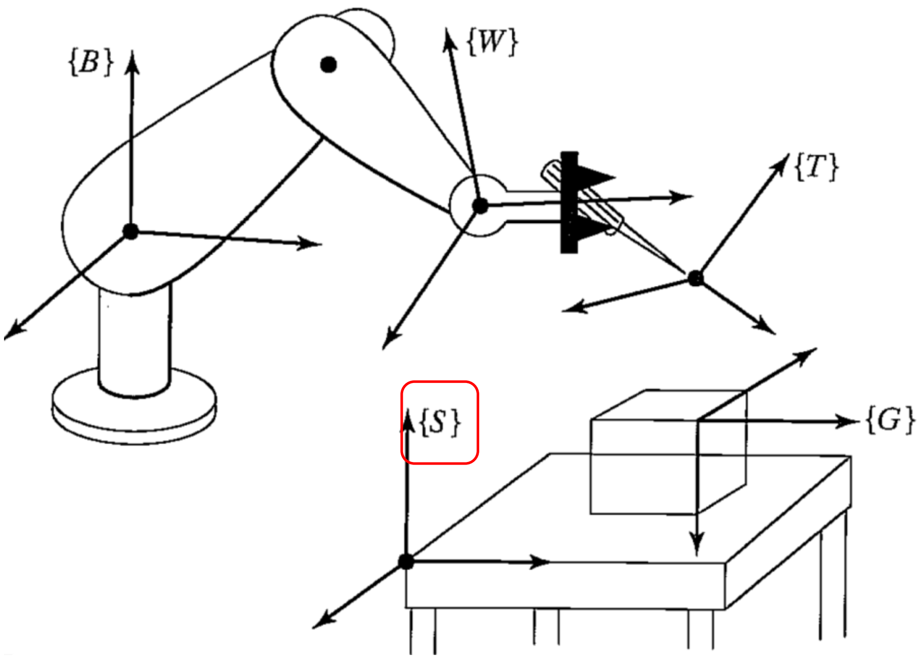


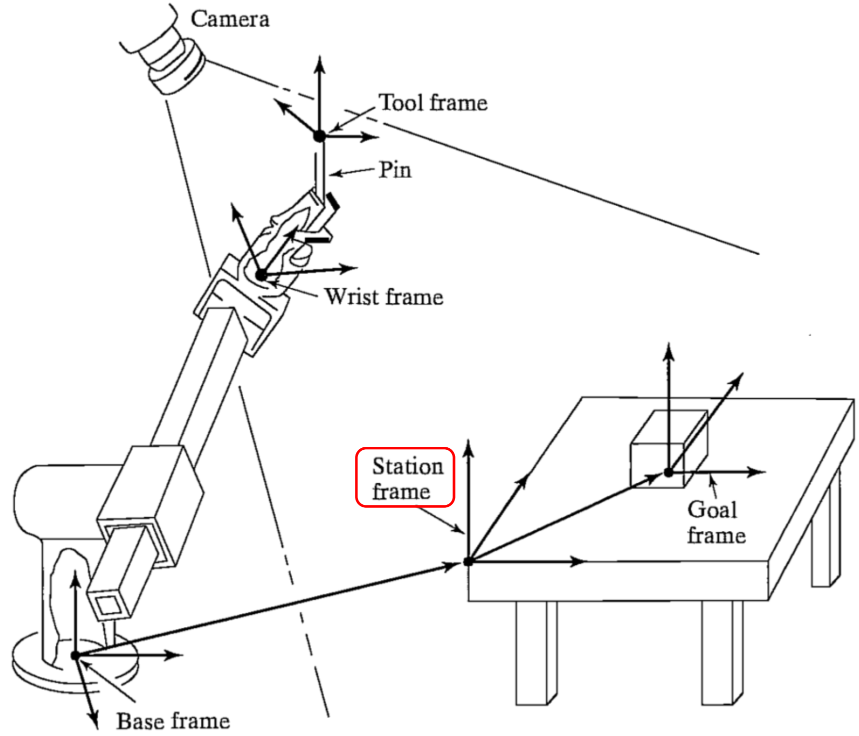


The Station Frame, {S}

Frames with Standard Names

- {S} is located in a task-relevant location. (Task/Work Frame/Universe Frame)
  - In the following figure, it is at the corner of a table upon which the robot is to work.
- As far as the user of this robot system is concerned, {S} is the universe frame, and all actions of the robot are performed relative to it.

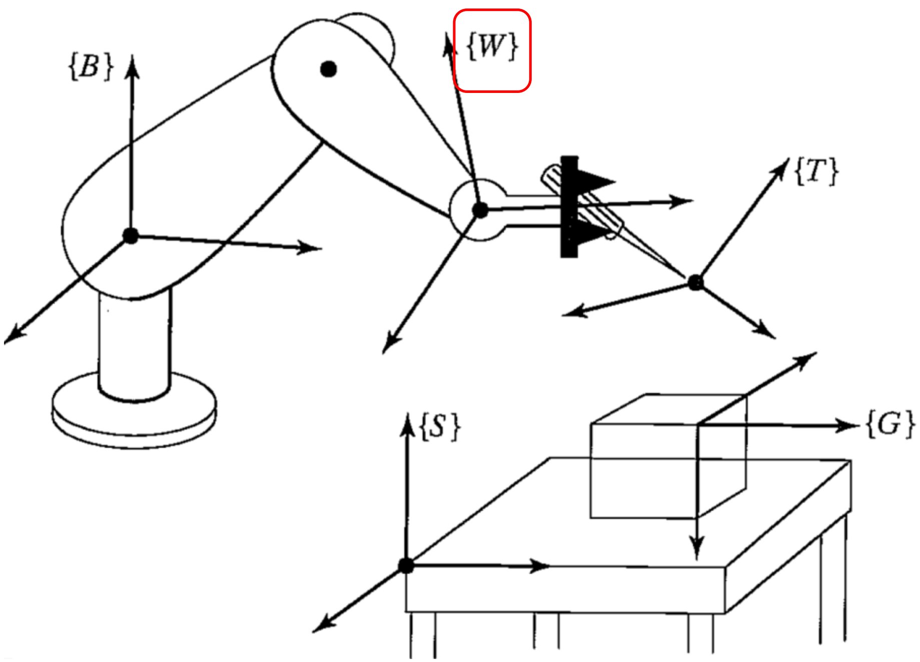


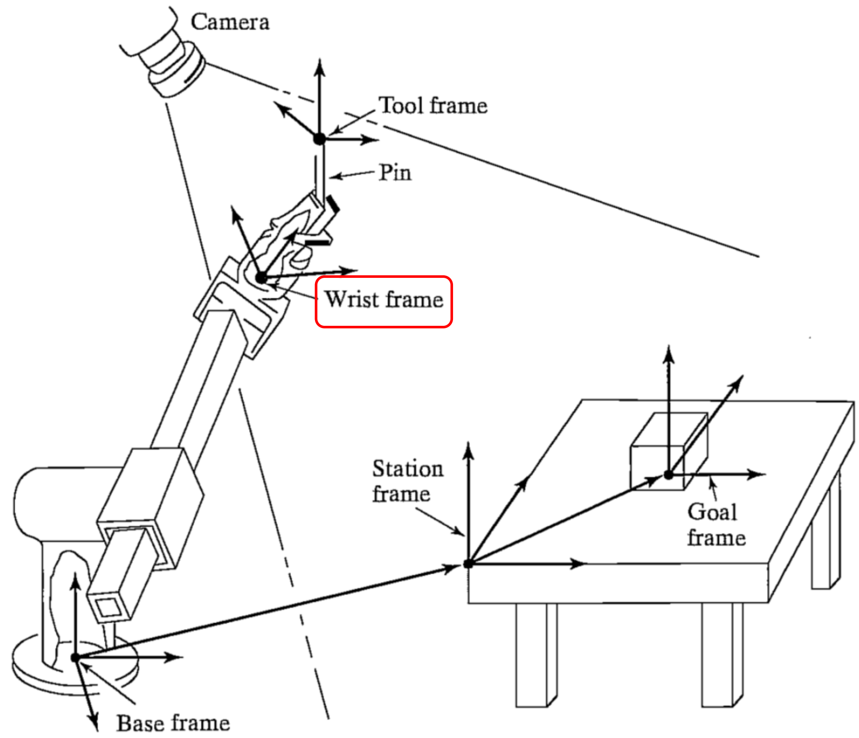


The Wrist Frame, {W}

Frames with Standard Names

- {W} is affixed to the last link of the manipulator.
    - It is another name for frame {N}, the link frame attached to the last link of the robot.
- Very often, {W} has its origin fixed at a point called the wrist of the manipulator, and {W} moves with the last link of the manipulator. It is defined relative to the base frame.

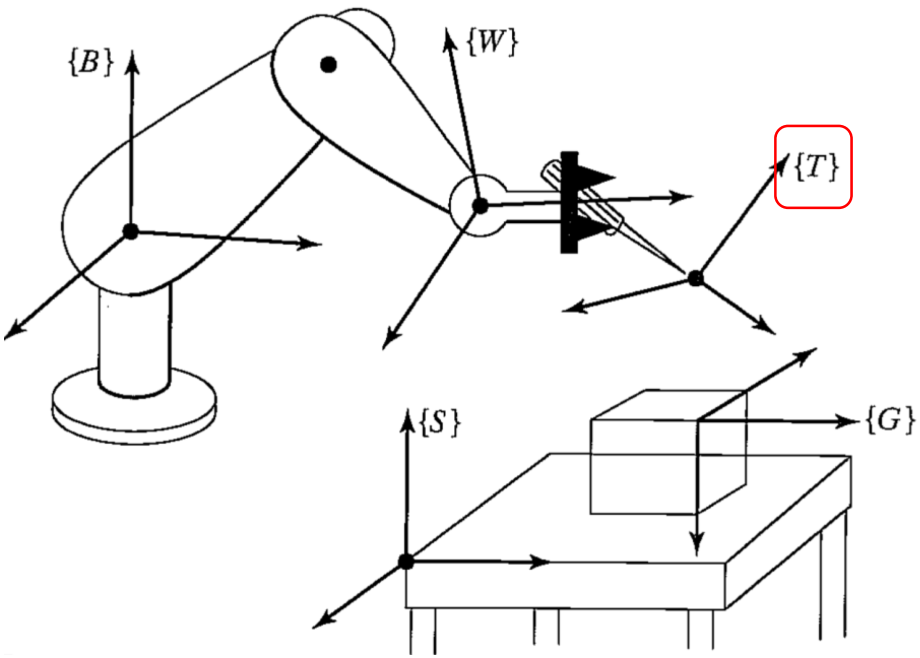


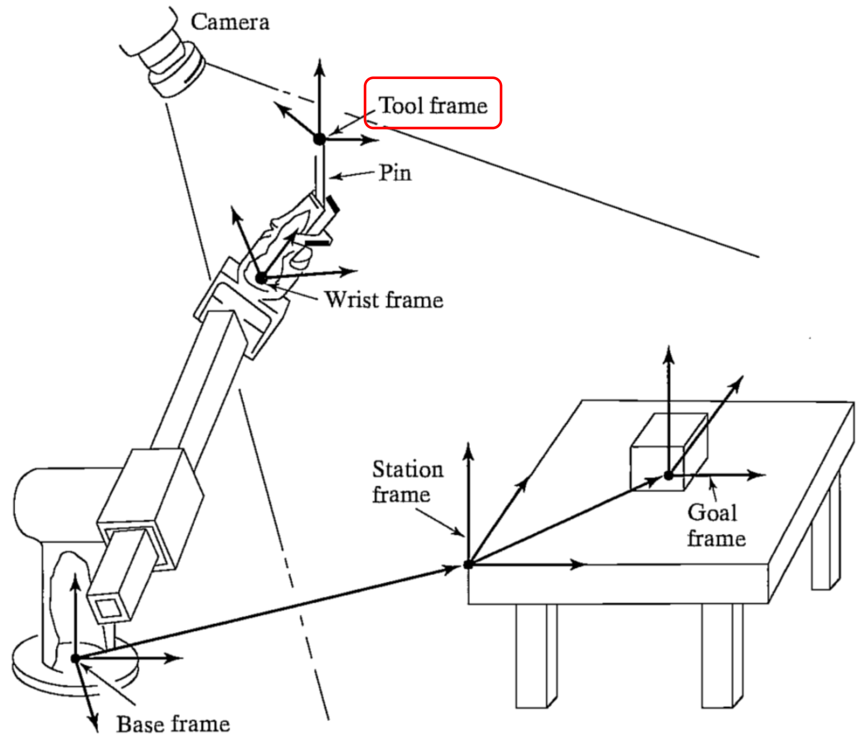


The Tool Frame, {T}

Frames with Standard Names

- {T} is affixed to the end of any tool the robot happens to be holding.
    - When the hand is empty, {T} is usually located with its origin between the robot fingertips.
- The tool frame is always specified with respect to the wrist frame.
    - In the following example, the tool frame is defined with its origin at the tip of a pin that the robot is holding.

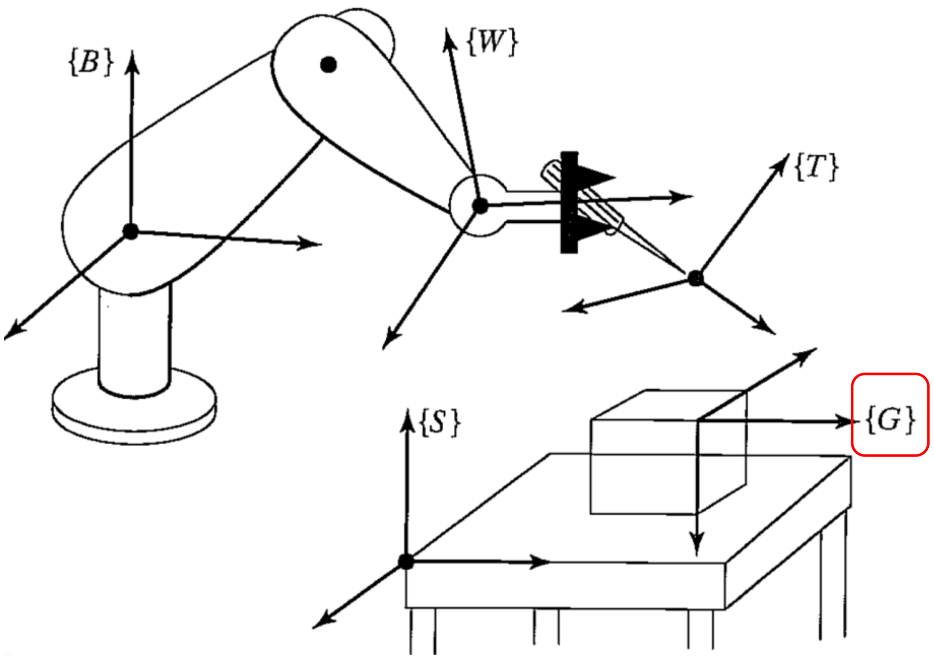


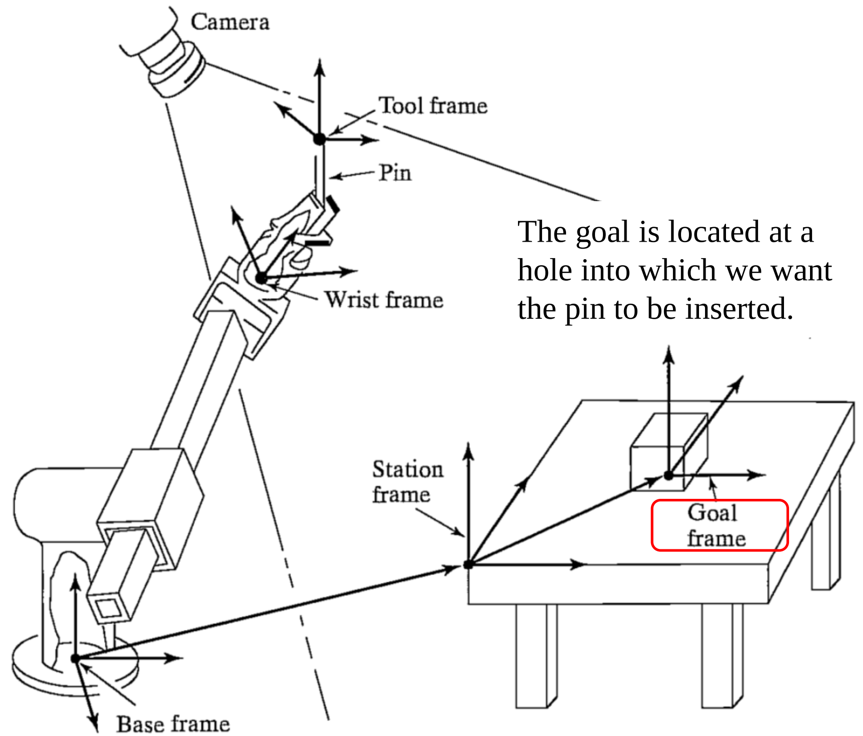


The Goal Frame, {G}

Frames with Standard Names

- {G} is a description of the location to which the robot is to move the tool.
  - Specifically this means that, at the end of the motion, the tool frame should be brought to coincidence with the goal frame.
- {G} is always specified relative to the station frame.





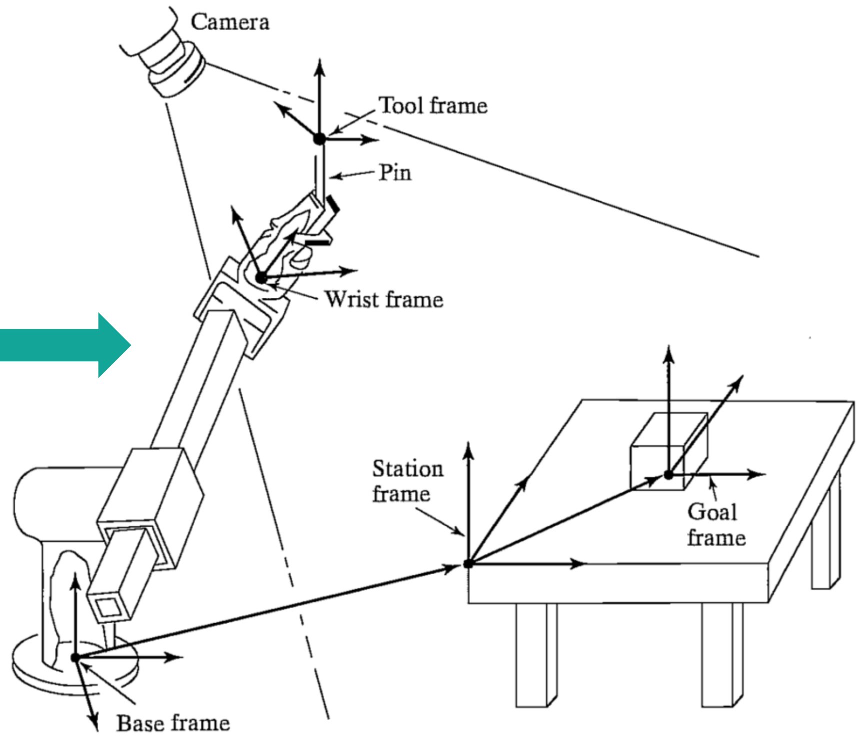
Where is the Tool?

We wish to calculate the value of the tool frame, {T}, relative to the station frame, {S}.

- The position and orientation of the tool it is holding (or of its empty hand) with respect to a convenient coordinate system.

$$ {}^{S}_{T}T = {}^{B}_{S}T^{-1}\, {}^{B}_{W}T\, {}^{W}_{T}T. $$

- **WHERE** function in some robot systems
  - It computes "where" the arm is.
  - The position and orientation of the pin relative to the table top



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Computation Considerations

In many practical manipulator systems, the time required to perform kinematic calculations is a consideration

- **The use of fixed- or floating-point representation of the quantities involved.**
    - Fixed over floating as the limited dynamic range of the variables
    - Roughly estimated a 24 fixed-point representation is enough.
- **Avoid computing common terms over and over throughout the computation**
    - Factoring equations of the transformation matrix to reduce the number of multiplications and additions at the cost of creating local variables (usually a good trade-off)
- **The calculation of the transcendental functions (sine and cosine) is a major expense in kinematics calculations**
    - Table-lookup implementations of the transcendental functions instead of actual calculations
- **Redundant computation of the kinematics as nine quantities are calculated to represent orientation**
    - One way is to calculate only two columns of the rotation matric and then to compute a cross product (requiring only six multiplications and three additions) to compute the third column.
    - Choose the two least complicated columns to compute.

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Thank you!

Prof. Song Chaoyang

- Dr. Wan Fang (sophie.fwan@hotmail.com)

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